

GENTZEN'S SECOND CONSISTENCY PROOF AND PREDICATIVE ANALYSIS

INAUGURAL-DISSERTATION

zur

Erlangung der Würde eines Doktors der Philosophie

vorgelegt der

Philosophisch-Naturwissenschaftlichen Fakultät

der Universität Basel

von

ROGER SCHOENBERGER

aus Basel

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Professor M.Eichler
Dekan



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1 INTRODUCTION

1.1 Historical remarks

The so-called second consistency proof is the technique developed by Gentzen in [6], where he proved the consistency of number theory by the principle of transfinite induction up to ϵ_0 . The method itself was soon forgotten since the interest in the consistency result alone was limited and other methods seemed to be more fruitful, such as the use of infinitary systems by Schütte [15] or Gödel's functional interpretation. But Scarpellini gave in [9], [11], [12], [13] several applications to problems about intuitionistic systems. In [10] he proposed a generalization of Gentzen's technique to systems where a formula can have a transfinite complexity. Based on this paper we give a treatment of predicative analysis developed by Feferman [3] and Schütte [17].

1.2 Organization of the paper

Parts of the text will be marked by sequences of numbers reflecting the logical dependence. References will be made by these numbers, possibly with a §-sign in front. Numbers in parentheses denote single formulas or cases and will be referred to by a #-sign in front if necessary. The main parts of the paper are §2 and §3 and the main results are presented in §3.5 and §3.6.

1.3 Outline of ideas

The purpose of §2 is to give a general frame for Gentzen's technique [6], based on ideas in Scarpellini's [10].

§2.1 is devoted to the generalization of the iterated exponential function $\omega_d(x)$ to transfinite d , following Bachmann [1]. This function plays an important role in [6] and [15] for the assignment of ordinals and its generalization allows us to use transfinite ordinals in the definition of the complexity of formulas. After introducing some notions concerning formal proofs in §2.2 we discuss general properties of ordinal assignment in §2.3 and of reduction theory in §2.4.

§3 gives a concise treatment of predicative analysis. In contrast to §2 we work here with a constructive system of ordinals, as discussed in §3.1.1. We construct our system of predicative analysis PA in §§3.1,3.2 according to the following idea. We start from a set parameter C and allow the definition of a hierarchy C^ω of sets based on C by transfinite induction. For the construction we use the Turing jump at successor ordinals and a collecting process at limit ordinals, as in the usual definition of the hyperarithmetical hierarchy. But the ordinals α which we allow in the construction are restricted by the "autonomy" condition that the well-ordering of $\{\beta \mid \beta < \alpha\}$ must have been proved in the theory TC_γ describing the hierarchy up to γ for some $\gamma < \alpha$. This procedure is quite similar to that of Feferman's IR [3].

In §§3.3, 3.4 we apply the methods of §2 to our theories TC_y .

§3.5 draws the conclusions of the reduction theory of §3.4.

In particular, we get a constructive consistency proof for our formal systems as well as a result on cut-elimination (using related infinitary systems, of course).

In §3.6 we show the equivalence of our system PA of predicative analysis with those of Feferman [3] and Schütte [17].

In particular we obtain the well-known characterization of predicative analysis by the ordinal Γ_0 . The fact that Γ_0 is a bound for the provable well-orderings of PA and that a number-theoretic theorem of PA can be proved in number theory by the principle of transfinite induction up to some ordinal $< \Gamma_0$ alone is a consequence of the reduction technique, while we use an interpretation of one of Feferman's systems into PA to show that Γ_0 is the least ordinal with this property.

1.4 Prerequisites

The general background of the paper is proof theory and recursion theory ([2], [8], [15]). The reader is assumed to have some knowledge of [3] and [6] and to be familiar with ordinal arithmetic as developed in [15].

2 GENTZEN'S TECHNIQUE

2.1 SOME ORDINAL FUNCTIONS

In this section, we construct a function η which shall be crucial in assigning ordinals to proofs. For the reduction theory, only its properties 2.1.12 are needed, so the reader may skip the rest. The construction is an application of Bachmann's results [1] §8.

2.1.1 Notation

Throughout 2.1, small Latin letters will range over ordinals of the second number class. The symbol $'$ stands for derivation, i.e. if ψ is a normal function, then ψ' denotes the normal function enumerating the fixed points of ψ . $\phi \circ \psi$ is the composition of the functions ϕ and ψ , i.e. $(\phi \circ \psi)(x) = \phi(\psi(x))$.

2.1.2 Definition

Following [1], we can define iterations $\langle n \rangle$ of a given normal function ψ by transfinite recursion, such that for every n , $\langle n \rangle$ is again a normal function:

- (1) $\langle 0 \rangle = \text{id}$ (id means identity : $\langle 0 \rangle(x) = x$)
- (2) $\langle n+1 \rangle = \langle n \rangle \circ \psi$
- (3) $\text{range } \langle m \rangle = \bigcap_{n \downarrow m} \text{range } \langle n \rangle$, if m is a limit ordinal.

We specify ψ by

- (4) $\psi(x) = \omega^x$.

2.1.3 Proposition

$$\langle n \rangle \circ \langle m \rangle = \langle n+m \rangle$$

Proof: [1] §8.

2.1.4 Proposition

If $n \triangleleft m$, then

- (1) $\text{range } \langle n \rangle \supsetneq \text{range } \langle m \rangle$ (strict inclusion)
- (2) $\langle n \rangle(x) \triangleleft \langle m \rangle(x)$
- (3) $\langle n \rangle(0) \triangleleft \langle m \rangle(0)$

Proof:

(1) follows immediately by induction.

There exists d such that $n+d=m$ and we have $\langle d \rangle(x) \triangleleft x$, since $\langle d \rangle$ is normal. So by 2.1.3 $\langle m \rangle(x) = \langle n+d \rangle(x) = \langle n \rangle(\langle d \rangle(x)) \triangleleft \langle n \rangle(x)$ which proves (2). Finally (3) follows by (2) and by $\langle n+1 \rangle(0) = \langle n \rangle(1) \triangleleft \langle n \rangle(0)$.

2.1.5 Proposition

$\langle n \cdot \omega \rangle = \langle n \rangle'$, in particular $\langle \omega \rangle(x) = \mathcal{E}_x$.

Proof:

It is enough to show that $\text{range } \langle n \cdot \omega \rangle = \text{range } \langle n \rangle'$.

Suppose first that $x \in \text{range } \langle n \rangle'$. Then $\langle n \cdot k \rangle(x) = x$ for any $k \triangleleft \omega$, since $\langle n \cdot k \rangle = \langle n+n+\dots+n \rangle = \langle n \rangle \circ \langle n \rangle \circ \dots \circ \langle n \rangle$ (k times) and x is a fixed point of $\langle n \rangle$. So $x \in \text{range } \langle m \rangle$ for any $m \triangleleft n \cdot \omega$ by 2.1.4#1, whence $x \in \text{range } \langle n \cdot \omega \rangle$.

Conversely, suppose $x \in \text{range } \langle n \cdot \omega \rangle$. Then $x = \langle n \cdot \omega \rangle(y)$ for some y and we have $\langle n \rangle \circ \langle n \cdot \omega \rangle(y) = \langle n+n \cdot \omega \rangle(y) = \langle n \cdot \omega \rangle(y)$. Hence x is a fixed point of $\langle n \rangle$; thus $x \in \text{range } \langle n \rangle'$.

2.1.6 Proposition

We have $\langle \omega^n \rangle(x) = \chi_x^{(n)}$, where $\chi^{(n)}$ is the sequence of normal functions defined in [3] as follows:

$$(1) \quad \chi_x^{(0)} = \omega^x$$

$$(2) \quad \chi^{(n+1)} = [\chi^{(n)}]'$$

$$(3) \quad \text{range } \chi^{(m)} = \bigcap_{n \leq m} \text{range } \chi^{(n)} \text{ for } m \text{ a limit ordinal}$$

Proof by induction on n :

$$n = 0 : \langle \omega^0 \rangle(x) = \langle 1 \rangle(x) = \omega^x = \chi_x^{(0)}$$

$$n = m+1 : \langle \omega^{m+1} \rangle = \langle \omega^m \rangle' \text{ by 2.1.5, } \langle \omega^m \rangle = \chi^{(m)} \text{ by the} \\ \text{induction hypothesis, } \chi^{(m)}' = \chi^{(m+1)} \text{ by definition,} \\ \text{hence } \langle \omega^{m+1} \rangle(x) = \chi_x^{(m+1)}.$$

n limit ordinal:

$$\text{Using the induction hypothesis, 2.1.4\#1, and} \\ \text{the fact that } \omega^x \text{ is a normal function, we get} \\ \text{range } \langle \omega^n \rangle = \bigcap_{m \leq \omega^n} \text{range } \langle m \rangle = \bigcap_{m \leq n} \text{range } \langle \omega^m \rangle = \\ = \bigcap_{m \leq n} \text{range } \chi^{(m)} = \text{range } \chi^{(n)}, \\ \text{so } \langle \omega^n \rangle = \chi^{(n)}.$$

2.1.7 Proposition

If $m, x \prec \Gamma_n$, then $\langle m \rangle(x) \prec \Gamma_n$, where $\Gamma_n = \phi^1(n)$ with $\phi(n) = \chi_0^{(n)}$.

Proof follows from 2.1.6, 2.1.4#2, and the analogous property of χ .

2.1.8 Proposition

Put $y(x) = \langle x \rangle(0)$. Then y is a normal function.

Proof:

Using 2.1.4#3, it remains to show that for a limit number z

$\langle z \rangle(0) = \sup_{x \leq z} \langle x \rangle(0)$. Now $\langle z \rangle(0) \geq \langle x \rangle(0)$ for all $x \leq z$,
so $\langle z \rangle(0) \leq \sup_{x \leq z} \langle x \rangle(0)$. By the normality of $\langle y \rangle$, $\sup_{x \leq z} \langle x \rangle(0)$
 $\in \text{range } \langle y \rangle$ for every $y \leq z$, whence $\langle z \rangle(0) \leq \sup_{x \leq z} \langle x \rangle(0)$.

Corollary: (1) $\gamma(x) \leq x$

(2) $\langle x \rangle(y) \leq \langle x+y \rangle(0)$

Proof: (1) is immediate, for (2) note that

$$\langle x \rangle(y) \leq \langle x \rangle(\gamma(y)) = \langle x \rangle(\langle y \rangle(0)) = \langle x+y \rangle(0).$$

2.1.9 Proposition

Γ enumerates the non-finite fixed points of γ , more

precisely: $\Gamma_x = \gamma'(2+x)$.

Proof:

It is easy to see that 0 and 1 are the only finite fixed points of γ . All others must be ε -numbers by 2.1.5, 2.1.4#1 and thus are fixed points of ϕ by 2.1.6. A similar argument shows that every fixed point of ϕ is also one of γ .

2.1.10 Proposition

$\langle n \rangle(x+1) \leq \langle n \rangle(x) \cdot \omega$ if $n \neq 0$.

Proof:

For $n=1$ we have $\langle 1 \rangle(x+1) = \omega^{x+1} = \omega^x \cdot \omega = \langle 1 \rangle(x) \cdot \omega$.

Now suppose $n \neq 1$.

By 2.1.4#1 $\text{range } \langle n \rangle \subset \text{range } \langle 1 \rangle$, so there exist y and z

such that $\langle n \rangle(x+1) = \langle 1 \rangle(z)$, $\langle n \rangle(x) = \langle 1 \rangle(y)$ and $y+1 \leq z$

by normality of the functions. Hence

$$\langle n \rangle(x+1) = \langle 1 \rangle(z) \leq \langle 1 \rangle(y+1) = \langle 1 \rangle(y) \cdot \omega = \langle n \rangle(x) \cdot \omega.$$

2.1.11 Definition

$$\eta(x, y, z) = \langle \delta(x, y) \rangle (z)$$

where δ denotes difference, i.e. $x + \delta(x, y) = y$ if $x < y$ and $\delta(x, y) = 0$ otherwise.

2.1.12 Properties of η :

- (1) For every pair u, v $\eta(u, v, x)$ is a normal function of x .
- (2) $v \preceq w \Rightarrow \eta(u, v, x) \preceq \eta(u, w, x)$
- (3) $v \preceq u \Rightarrow \eta(u, v, x) = x$
- (4) $u \preceq v \preceq w \Rightarrow \eta(u, v, \eta(v, w, x)) = \eta(u, w, x)$
- (5) $u < v \Rightarrow \eta(u, v, x) \cdot \omega^d \preceq \eta(u, v, x + d)$
- (6) $u < v; x_1, \dots, x_n < z \Rightarrow \bigoplus_{i=1}^n \eta(u, v, x_i) < \eta(u, v, z)$
- (7) $u, v, x < \Gamma_0 \Rightarrow \eta(u, v, x) < \Gamma_0$
- (8) $\eta(u, v, x) \preceq \eta(0, v + x, 0)$

Proof:

All properties are immediate consequences of the definitions 2.1.2 and 2.1.11. (2) follows from 2.1.4#2 and (4) from 2.1.3. (5) is proved by induction on d using 2.1.10 and normality properties of the functions occurring in it. (6) follows from (5) for $d=1$ and the property $\bigoplus_{i=1}^n x < x \cdot \omega$ of the natural sum $\#$. Finally (7) follows from 2.1.7, and (8) from 2.1.8#2.

2.1.13 Remark

$\eta(0, d, x)$ generalizes the function $\omega_d(x)$ used in [6] and [15] to transfinite d .

2.2 PROOF TREES

In this section we describe some notions concerning proofs. It is clear that everything can be coded within the natural numbers, but for ease of reading we give the definitions in an intuitive way.

2.2.1 Rules

We assume that we are given a language defining a set of sequents. Certain operations allow us to deduce certain sequents from others: We distinguish axioms (no premisses), n-rules (finite number n of premisses), and ω -rules (infinite number of premisses, given by a recursive function). Certain 1-rules not affecting the logical structure of the formulas in the sequents are called weak. The cut is a 2-rule of the form

$$\frac{\Gamma \rightarrow \Delta, A \quad A, \Pi \rightarrow \Sigma}{\Gamma \Pi \rightarrow \Delta \Sigma}$$

with A the cut-formula, and is always assumed to be present.

All other rules are called strong rules.

For an example of these notions, see 3.2.

2.2.2 Derivations and proofs

The basis for a derivation is a countable well-founded tree, intuitively upward branching, with lowest element. Between the elements (or nodes) of the tree natural relations are defined such as: N is below N' (written $N \sqsubset N'$), N is a predecessor of N', N is topmost (i.e. has no predecessors).

If N is a non-topmost node then N_i denotes the $(i+1)$ -st predecessor of N counted from "left to right". The bottom node is always denoted by E .

Now a derivation D is given by a tree together with a function assigning to every node N a figure $F_D(N)$ of the form described below. (When no confusion may arise we shall also write $F(N)$ or simply F for $F_D(N)$, and the same convention will be used for S and R introduced below.)

If N is topmost then $F(N)$ is of the form

$$\frac{S'}{S}$$

where S and S' are sequents and the double bar indicates a series of applications of weak rules by which S follows from S' . S' is called an assumption of the derivation.

If N is not topmost then $F(N)$ is of the form

$$\frac{S_i, 0 \leq i < n}{\frac{S'}{S}} R$$

where $S_i = S(N_i)$ for $0 \leq i < n$, R is a strong rule or a cut, called the rule at N or $R(N)$, n is the number of predecessors of N equal to the number of premisses of R (we include the case $n=0$), S' follows from S_i ($0 \leq i < n$) by R , and S , the sequent at N or $S(N)$, follows from S' by weak rules.

$D \upharpoonright N$ denotes the restriction of D to the nodes $M \sqsubseteq N$, so

$S_{D \upharpoonright N}(E) = S_D(N)$. $S_D(E)$, the sequent at the bottom node, is called the end-sequent of D . The derivation is called a proof if all assumptions are axioms.

2.2.3 Definition

A strong rule occurring at a node N of a derivation D is called critical, if no other strong rule occurs at a node below N in P. The final part D^- of D is the restriction of D containing the bottom node E, and with every node N such that $R(N)$ is a cut, also the two predecessors. Thus only critical rules or assumptions occur at the top-most nodes of D^- .

2.3 ASSIGNMENT OF ORDINALS TO PROOFS

It seems natural to assign ordinals inductively over the proof trees using monotonic functions (cf. §2.3.1) to compute the ordinal of a node from that of its predecessors. But we have to increase the ordinal further by means of the function η (cf. §2.1) at those nodes which have greater "cut complexity" (cf. §2.3.2) than the nodes below, to ensure that certain "reductions" applied to proofs, which apparently complicate the proofs, actually lower the ordinal assigned (cf. §2.3.3).

2.3.1 Inference function

To every n-rule R we assign its inference function ζ_R with n arguments such that

- (1) if $x_i \leq y_i$ for $1 \leq i \leq n$, then $\zeta(x_1, \dots, x_n) \leq \zeta(y_1, \dots, y_n)$
- (2) if $x_i \neq 0$ for $1 \leq i \leq n$, then $\zeta(x_1, \dots, x_n) \neq 0$
- (3) if $x_i \neq 0$ for $1 \leq i \leq n$,
then $\zeta(\eta(u, v, x_1), \dots, \eta(u, v, x_n)) \leq \eta(u, v, \zeta(x_1, \dots, x_n))$

Example:

In case of a cut-rule, we shall always take

$$\zeta_{\text{cut}}(x_1, x_2) = x_1 \# x_2.$$

2.3.2 Complexity and height

To every node N of a derivation D we assign its complexity $c_D(N)$, depending on $F_D(N)$. If $R_D(N)$ is a cut, then $c_D(N)$ will be the complexity of the cut-formula A (to be defined for a specific system), and we set $c_D(N)=0$ if N has infinitely many predecessors. The height $h_D(N)$ is the maximum of $c_D(M)$ for $M \sqsubset N$, and $h_D(E)=0$.

2.3.3 Assignment of ordinals to the nodes of a derivation

Let D be a derivation, c a complexity assignment for D and h the corresponding height (cf. §2.3.2), f a function assigning an ordinal $\neq 0$ to the topmost nodes of D . Then we define inductively over the tree of D a function assigning an ordinal $o(D, c, f, N)$ to every node N of D , as follows:

- (1) if N is topmost, then $o(D, c, f, N)=f(N)$
- (2) if N has infinitely many predecessors N_i , then

$$o(D, c, f, N) = \sup_{i < \omega} o(D, c, f, N_i)$$
- (3) if N has a finite number $n+1$ of predecessors $N_0, \dots, N_n$, then

$$o(D, c, f, N) = \eta(h(N), c(N), o^-(N)),$$
 where

$$o^-(N) = \zeta_{R(N)}(o(D, c, f, N_0), \dots, o(D, c, f, N_n)).$$

$$\eta$$
 is given by §2.1.11 and ζ by 2.3.1.

When the context allows no confusion, we shall abbreviate $o(D, c, f, N)$ to $o(D, N)$ or $o_D(N)$, or even to $o(N)$.

Note: $\sup_{i < \omega} \xi_i$ denotes the least ordinal $> \xi_i$ for all i .

2.3.4 Ordinal of a proof

Assume that g is a function assigning an ordinal $\neq 0$ to the axioms of the system considered and put $f_P(N) = g(S'_P(N))$ for N topmost. Then we define the ordinal of a proof P by $O(P) = o(P, c_P, f_P, E)$ (cf. §2.3.3). g and c will be kept fixed for a given system.

2.3.5 Tree ordinal

The tree ordinal $T(D)$ of a derivation is an ordinal naturally connected with the tree of the derivation and is defined inductively as follows:

- (1) if N is topmost $t(D, N) = 1$
- (2) otherwise $t(D, N)$ is the least ordinal $\succ t(D, M)$
for all predecessors M of N
- (3) $T(D) = t(D, E)$.

2.4 REDUCTION THEORY

In the present section we will discuss some generalities on reduction theory and mention three basic lemmas.

2.4.1

Reductions are certain transformations on proofs such that their application lowers the ordinal assigned. A proof is irreducible (or in normal form) if no reduction can be applied to it. A proof P reduces to P' if there is a sequence of reductions which transforms P successively into P' .

2.4.2

For every proof P it follows by means of the principle of transfinite induction up to $O(P)$ that there is a proof P' in normal form such that P reduces to P' , since there cannot be an infinite descending sequence of ordinals. In contrast to the situation with natural deduction, P' is not unique in general.

2.4.3 Lemma

Suppose $f(N) \preceq f'(N)$ for the topmost nodes N of a derivation D . Then $o(D, c, f, N) \preceq o(D, c, f', N)$ for all nodes N of D and an arbitrary function c .

Proof by induction on the tree of D using 2.3.1#1 and 2.1.12#1.

2.4.4 Lemma

Assume that N_0 is a topmost node of D which is in the final part D^- and that $f(N) = f'(N)$ for $N \neq N_0$ whereas $f(N_0) \prec f'(N_0)$. Then $o(D, c, f, E) \prec o(D, c, f', E)$.

Proof by induction on the path from N_0 to E using 2.3.12#1 and the fact that only the strictly monotonic inference function $\zeta_{\text{cut}}(x, y) = x \# y$ comes to effect below N_0 .

2.4.5 Lemma

If $c(N) \preceq c'(N)$ for all nodes N of D , then

$o(D, c, f, E) \preceq o(D, c', f, E)$.

Proof:

Let h and h' denote the heights corresponding to c and c' .

Clearly $h(N) \preceq h'(N)$. The theorem is proved when we have shown

that $o(D, c, f, N) \leq \eta(h(N), h'(N), o(D, c', f, N))$, since $h(E) = h'(E) = 0$.

But this may be done by induction on the tree of D using

2.3.1#3, 2.3.3, 2.1.12##1, 3, 4, as follows.

If N is topmost, then the statement is immediate. Now assume

that N has exactly one predecessor (denoted by N_0). Put

$\underline{h}(N) = h(N_0)$, $\underline{h}'(N) = h'(N_0)$, $o'(N) = o(D, c', f, N)$. By the

induction hypothesis we get

$$\begin{aligned} o(D, c, f, N) &= \eta(h(N), \underline{h}(N), \zeta_{R(N)}(o(D, c, f, N_0))) \leq \\ &\eta(h(N), \underline{h}(N), \zeta(\eta(\underline{h}(N), \underline{h}'(N), o'(N_0)))) \leq \\ &\eta(h(N), \underline{h}(N), \eta(\underline{h}(N), \underline{h}'(N), \zeta(o'(N_0)))) = \eta(h(N), \underline{h}'(N), \zeta(o'(N_0))) = \\ &\eta(h(N), h'(N), \eta(h'(N), \underline{h}'(N), \zeta(o'(N_0)))) = \eta(h(N), h'(N), o'(N)). \end{aligned}$$

The same argument works of course for all nodes with finitely

many predecessors. If N has infinitely many predecessors,

we get $h(N_i) = h(N)$ and $h'(N_i) = h'(N)$ since by definition $c(N) = 0$.

Hence by the induction hypothesis $o(D, c, f, N) = \sup_{i < \omega} o(D, c, f, N_i) \leq$

$$\begin{aligned} \sup_{i < \omega} \eta(h(N_i), h'(N_i), o'(N_i)) &= \eta(h(N), h'(N), \sup_{i < \omega} o'(N_i)) = \\ &\eta(h(N), h'(N), o'(N)) . \end{aligned}$$

2.4.6 Remark

§2.4.4 shows that a reduction is O.K. if it lowers the

ordinal somewhere in the final part and §§2.4.3, 2.4.5

ensure that certain operations do no harm.

3 PREDICATIVE ANALYSIS

3.1 SETS DEFINED BY TRANSFINITE INDUCTION

We construct a hierarchy of sets of natural numbers using a constructive ordinal system.

3.1.1

We assume that we are given a well-ordering, defined on a subset D of the natural numbers, such that the following relations, operations and functions are recursive.

- (1) Relations $x \in D$, $x \prec y$, $\text{Lim}(x)$, $\text{Suc}(x)$ with their obvious meaning.
- (2) Operations $\oplus$ (ordinal sum), $\otimes$ (ordinal multiplication) and $\#$ (natural sum).

- (3) Functions $\omega(x)$ representing exponentiation ω^x and $\eta(u, v, x)$ with properties analogous to §2.1.12 (in particular $\#5$ has to be rewritten as

$$u \prec v \Rightarrow \eta(u, v, x) \otimes \omega(d) \preceq \eta(u, v, x \oplus d)).$$

Thus a notation is assigned to every ordinal $\prec \Gamma_0$.

- (4) A function $\hat{x}$ which enumerates the finite ordinals $(\hat{0}, \hat{1}, \dots)$. Their limit is denoted by ω , as usual.

- (5) Functions φ, ∂, π with the following properties:

$$(5.1) \quad \text{Lim}(x) \Rightarrow \varphi(n, x) \prec \varphi(n+1, x) \prec x$$

$$(5.2) \quad \text{Lim}(x), y \prec x \Rightarrow y \prec \varphi(\partial(y, x), x)$$

$$(5.3) \quad \text{Suc}(x) \Rightarrow \pi(x) \oplus \hat{1} = x, \quad \neg \text{Suc}(x) \Rightarrow \pi(x) = x$$

$$(5.4) \quad \text{Lim}(x) \Rightarrow \text{Suc}(\varphi(n, x))$$

$$(5.5) \quad \neg \text{Lim}(x) \Rightarrow \varphi(n, x) = x$$

φ gives fundamental sequences and π the predecessors.

We can use the constructive ordinal system of Feferman [4] or Schütte [17] for our purpose here. Note that since equality between ordinals is only represented as a (recursive) equivalence relation $\equiv$ between numbers there, we have first to pass to a system of unique notations by means of the μ -operator as indicated in [4], or replace $\equiv$ between ordinals by $\equiv$ in the subsequent development, which would not change the proof-theoretic argument.

To see that our function η can be defined constructively, note first that we shall use it later (§3.3.1) only where the arguments u, v are in Cantor normal form, i.e. can be written as $\omega(\alpha_1) \oplus \dots \oplus \omega(\alpha_n)$ with $\alpha_1 \leq \dots \leq \alpha_n$. Of course, the definition is then general since every ordinal $\neq \hat{0}$ can be written in this form (with α_i computable from α). If ν and μ are in Cantor normal form, we can write $\nu = \bigoplus_{i=1}^k \bigoplus_{j=1}^{n_i} \omega(\alpha_i)$, $\mu = \bigoplus_{i=1}^k \bigoplus_{j=1}^{m_i} \omega(\alpha_i)$, where $\alpha_1 \leq \dots \leq \alpha_k$, $\bigoplus_{i=1}^0 \beta_i = \hat{0}$, $\bigoplus_{i=1}^{k+1} \beta_i = (\bigoplus_{i=1}^k \beta_i) \oplus \beta_{k+1}$. Then $\nu \# \mu = \bigoplus_{i=1}^k \bigoplus_{j=1}^{n_i+m_i} \omega(\alpha_i)$. The difference of two such numbers can be written $\delta(\nu, \mu) = \hat{0}$ if $\mu \leq \nu$ and otherwise $\delta(\nu, \mu) = \bigoplus_{i=1}^k \bigoplus_{j=1}^{d_i} \omega(\alpha_i)$, where $d_i = 0$ for $1 \leq i < l$, $d_l = m_l - n_l$, $d_i = m_i$ for $l < i \leq k$ with l such that $m_i = n_i$ for $1 \leq i < l$ and $m_l > n_l$. Hence $\delta(\nu, \mu)$ is again in Cantor normal form if we include the empty string with value $\hat{0}$. Let $\kappa(\alpha, x)$ be the constructive function representing the classical $\chi_x^{(\alpha)}$. Let $\tilde{\kappa}(\alpha, x)$ for α in Cantor normal form be defined by induction on the number of terms in α , thus:
 $\tilde{\kappa}(\hat{0}, x) = x$, $\tilde{\kappa}(\beta \oplus \omega(\gamma), x) = \tilde{\kappa}(\beta, \kappa(\gamma, x))$. Now the function η can be defined by $\eta(u, v, x) = \tilde{\kappa}(\delta(u, v), x)$ (use §§2.1.6, 2.1.11, 2.1.3).

3.1.2 Definition

By clauses (1),(2),(3) below, we define a transfinite sequence C^n of sets starting from any set C of natural numbers. We use the constructive system of ordinals 3.1.1, with functions and relations defined there. C_e^n denotes the set recursively enumerable in C^n with Gödel number e , ϑ, π are given by 3.1.1#5, and $\langle x, y \rangle = \frac{1}{2}(x+y)(x+y+1)+y$ is a recursive pairing function.

$$(1) \quad C^{\hat{0}} = C$$

$$(2) \quad \langle x, e \rangle \in C_e^m \iff x \in C_e^{\pi(m)} \quad \text{if } \text{Suc}(m)$$

$$(3) \quad \langle x, n \rangle \in C_e^m \iff x \in C_e^{\vartheta(n, m)} \quad \text{if } \text{Lim}(m)$$

Thus $\{\emptyset^n, n \in D\}$ represents a segment of the hyperarithmetical hierarchy (cf. [2] §11.5). We shall say that a set is of level α or in conventional notation $\Sigma_1(C^\alpha)$, if it is C_e^α for some e .

3.1.3

There exist partial recursive functions $\phi_i, 1 \leq i \leq 9$, such that

$$(1) \quad \psi(x) \in C_e^m \iff x \in C_{\phi_1(\ulcorner \psi \urcorner, e)}^m$$

$$(2) \quad \langle x, y \rangle \in C_e^m \iff x \in C_{\phi_2(e, y)}^m$$

$$(3) \quad x \in C_e^m \iff x \in C_{\phi_3(e, m, n)}^n \quad \text{if } m \leq n$$

$$(4) \quad x \in (C_e^m)^n \iff x \in C_{\phi_4(e, m, n)}^{m \oplus n}$$

$$(5) \quad x \in C_e^m \wedge y \in C_f^m \iff \langle x, y \rangle \in C_{\phi_5(e, f)}^m$$

$$(6) \quad x \in C_e^m \vee y \in C_f^m \iff \langle x, y \rangle \in C_{\phi_6(e, f)}^m$$

$$(7) \quad \neg x \in C_e^m \iff x \in C_{\phi_7(e)}^{m \oplus \hat{1}}$$

$$(8) \quad \exists y (x \in C_{\psi(y, z)}^m) \iff x \in C_{\phi_8(\ulcorner \psi \urcorner, z)}^m$$

$$(9) \quad \forall y (x \in C_{\psi(y, z)}^m) \iff x \in C_{\phi_9(\ulcorner \psi \urcorner, z)}^{m \oplus \hat{2}}$$

Furthermore there exist constants $\phi_0, \phi_=_$ such that

$$(10) \quad C_{\phi_0}^n = C, \quad x = y \iff \langle x, y \rangle \in C_{\phi_=_}^n.$$

Notation: m, n are variables for constructive ordinals (§3.1.1)

and ψ for recursive functions (with Gödel number $\ulcorner \psi \urcorner$).

Remark

We refer the reader to [2], [9] (esp. §§5.5, 14, 5). Apart from (3) and (4) it is quite easy to define the functions ϕ_i intuitively by interpreting $x \in C_e^n$ as "the (Turing) program with number e using an oracle for C^n terminates on input x ". Note that these cases are implicit in the general theorem that if a set is $\Sigma_{k+1}(C^n)$ then it is $\Sigma_1(C^{n \oplus k})$ uniformly (which follows from [9]§14.5). Since (4) is not used later, we give only the

Proof of (3).

We define ϕ_3 using Kleene's recursion theorem.

$$(3.1) \quad \phi_3(e, m, n) = 0 \quad \text{for } m \not\leq n \text{ or } m \notin D \text{ or } n \notin D$$

$$(3.2) \quad \phi_3(e, m, n) = e \quad \text{for } m = n$$

(3.3) If $m \leq n$ and $\text{Suc}(n)$ we have

$$x \in C_e^{\pi(n)} \iff \langle x, e \rangle \in C^n \iff x \in C_{\phi_2(\phi_0, e)}^n.$$

$$\text{So we take } \phi_3(e, m, n) = \phi_2(\phi_0, \phi_3(e, m, \pi(n))).$$

(3.4) If $m \leq n$ and $\text{Lim}(n)$ we have

$$m \leq \psi(m, n) \leq n \text{ for } \psi(m, n) = \pi(\varphi(\sigma(m, n), n)),$$

$$\text{so } x \in C_e^m \iff x \in C_{\phi_3(e, m, \psi(m, n))}^{\psi(m, n)} \text{ can be used.}$$

$$\text{Now } \langle \langle x, y \rangle, z \rangle \in C^n \iff \langle x, y \rangle \in C^{\varphi(z, n)} \iff x \in C_y^{\pi(\varphi(z, n))}$$

$$\text{and } \langle \langle x, y \rangle, z \rangle \in C^n \iff x \in C_{\phi_2(\phi_0, \phi_2(\phi_0, z), y)}^n, \text{ so we can}$$

$$\text{set } \phi_3(e, m, n) = \phi_2(\phi_2(\phi_0, \sigma(m, n)), \phi_3(e, m, \psi(m, n)))$$

Finally, we can show by induction on n that ϕ_3 is everywhere defined and has property (3).

3.2 FORMAL SYSTEMS TC AND PA

In this section we shall define a first-order formal system in which we can develop the theory of §3.1. In §3.2.14 we give a system of predicative analysis.

3.2.1 The Hierarchy TC of Systems

We wish TC_α for any $\alpha \in D$ to describe the hierarchy 3.1.2 up to α . C acts as a parameter, which is essential for expressing a general principle of transfinite induction by a formula. The language of the theory is given by 3.2.2 to 3.2.4, the axioms by 3.2.6 to 3.2.8, and the rules by 3.2.9 to 3.2.12. Only the rules 3.2.11 and 3.2.12 depend on α . Clearly, TC_α is a subsystem of TC_β for $\alpha < \beta$. TC denotes the collection of the theories TC_α for $\alpha \in D$.

3.2.2 Terms

As usual, terms are constructed from constants and variables by means of function symbols. If t is a constant term, i.e. it contains no variables, then $|t|$ denotes its value.

Constants: 0 (We shall introduce new constants $1, 2, \dots$, and little greek symbols for ordinal notations as abbreviations for the corresponding constant terms.)

Variables: $x_1, x_2, x_3, \dots$ ranging over the natural numbers.

Function symbols: $\Psi_1, \Psi_2, \Psi_3, \dots$ representing a list of recursive functions containing at least the primitive recursive ones. (We use the informal operations and functions $'$, $+$, $\cdot$, and those of 3.1.1 and 3.1.3 as abbreviations for their formal counterparts. $\langle x, y \rangle$ stands for a bijective mapping $N^2 \rightarrow N$ (see 3.1.2) and $(x)_0, (x)_1$ denote the inverse functions, such that $\langle (x)_0, (x)_1 \rangle = x$.)

We use w, x, y, z as meta-variables for formal variables and s, t, u, v for terms.

3.2.3 Formulas

If s and t are terms, then $s=t$ and $s \neq t$ are prime formulas, where $s=t$ expresses equality and $s \neq t$ expresses the informal notion $s \in C^t$ of §3.1.2. We use the relations in §3.1.1#1 as abbreviations for prime formulas of the form $t=0$ with suitable terms t , thus $x < y$ e.g. is an abbreviation for the formula $\Psi(x, y)=0$ with a certain recursive function Ψ . Formulas are constructed from prime formulas as usual by means of the logical symbols $\neg, \wedge, \vee, \supset, \forall, \exists$. A, B, F, G are meta-variables for formulas. We use $F \leftrightarrow G$ as an abbreviation for $(F \supset G) \wedge (G \supset F)$ and $\forall x \prec y F$ for $(\forall x)(x \prec y \supset F)$. We shall also write $x \in C$ for $x \in \hat{0}$.

The general principle of transfinite induction up to β is expressed by $\forall x \in D (\forall y \prec x y \in C \supset x \in C) \supset \forall x \prec \beta x \in C$ which we shall abbreviate as $I(\beta)$. The relation $x \in C_{\hat{0}}^n$ can be expressed by means of Kleene's T-predicate. A formula in which $s \neq t$ occurs only if t is constant and $|t| \leq \beta$ is called a β -formula, so if n is constant, then $x \in C_{\hat{0}}^n$ is a $|n|$ -formula.

3.2.4 Sequents

Sequents are expressions of the form $\Gamma \rightarrow \Delta$, where Γ and Δ are finite, possibly empty, sequences of formulas. A sequent is saturated if no free variables occur in its formulas. A sequent containing only formulas of the form $s=t$ and at most one formula in the succedent is called simple.

3.2.5 Definition of $\sim$

We call two formulas A, B isomorphic, and write $A \sim B$, if B follows from A by application of a finite (possibly empty) series of the following operations (where s, t, u are constant terms):

- (1) replacement of a term s by t , if $|s| = |t|$.
- (2) replacement of a prime formula $\langle s, t \rangle \in u$ by $s \in \varphi(t, u)$ or conversely, where $\text{Lim}(|u|)$.

$\sim$ is an equivalence relation.

3.2.6 Logical Axioms

are of the form $A \rightarrow B$, where A and B are isomorphic formulas.

3.2.7 Mathematical Axioms

- (1) all saturated simple sequents which are numerically true
- (2) a set of simple sequents including equality axioms, recursion equations, and sequents expressing properties of our well-ordering D (§3.1.1).

3.2.8 C-Axioms

- (1) $s=u, t=v, s \in t \rightarrow u \in v$
- (2) $\text{Lim}(v), \langle s, t \rangle \in v \rightarrow s \in \varphi(t, v)$
- (3) $\text{Lim}(v), s \in \varphi(t, v) \rightarrow \langle s, t \rangle \in v$

#1 is an equality axiom, #2 and #3 formalize §3.1.2#3.

Note the restricted form of the axioms in §3.6.1.

3.2.9 Weak Rules

- (1) the usual structural rules: thinning, contraction, interchange
- (2) conversion:
$$\frac{A_1, \dots, A_n \rightarrow B_1, \dots, B_m}{A'_1, \dots, A'_n \rightarrow B'_1, \dots, B'_m}$$

where $A_i \sim A'_i$, $B_j \sim B'_j$, $1 \leq i \leq n$, $1 \leq j \leq m$.

3.2.10 Strong Rules

- (1) The usual first order rules $\rightarrow S$ and $S \rightarrow$ for every logical symbol S
- (2) induction rules 3.2.11
- (3) C-rules 3.2.12

3.2.11 Induction Rules

- (1) complete induction (I) :
$$\frac{\Gamma, A(z) \rightarrow A(z'), \Delta}{\Gamma, A(0) \rightarrow A(t), \Delta}$$
- (2) transfinite induction up to β (TI_β) for any constant $\beta \in D$ with $\hat{1} \leq \beta \leq \alpha$ (§3.1.1) :

$$TI_\beta : \frac{z < \beta, \forall x < z A(x), \Gamma \rightarrow \Delta, A(z)}{t < \beta, \Gamma \rightarrow \Delta, A(t)}$$

where z is a variable not occurring in the conclusion,
 t a term free for x in A , A the induction formula.

3.2.12 C-Rules

$$\rightarrow C_\alpha \quad \frac{\Gamma \rightarrow \Delta, s \in^* t}{t < \alpha, \Gamma \rightarrow \Delta, s \in t \hat{1}}$$

$$C \rightarrow_\alpha \quad \frac{s \in^* t, \Gamma \rightarrow \Delta}{t < \alpha, s \in t \hat{1}, \Gamma \rightarrow \Delta}$$

where $s \in^* t$ is an abbreviation for the formula expressing $(s)_0 \in C_{(s)_1}^t$.

The C-rules formalize §2.1.2#2.

3.2.13 Autonomous Ordinals

We define the set of autonomous ordinals ACD by:

- (1) $\hat{0} \in A$
- (2) If $\beta \in A$ and $TC_\beta \vdash \rightarrow I(\gamma)$, then $\gamma \in A$.

3.2.14 System of Predicative Analysis

We choose our system of predicative analysis PA to be $PA = \bigcup_{\alpha \in A} TC_\alpha$, where A is given by 3.2.13.

Alternatively, we may describe PA as follows.

PA has the same language and axioms as TC , whereas we change the rules which depend on α , by adding $\rightarrow I(\alpha)$ as a premise, as follows:

$$TI \quad \frac{\rightarrow I(\alpha) ; z < \alpha, (\forall x < z) F(x), \Gamma \rightarrow \Delta, F(z)}{t < \alpha, \Gamma \rightarrow \Delta, F(t)}$$

$$\rightarrow C \quad \frac{\rightarrow I(\alpha) ; \Gamma \rightarrow \Delta, s \in^* t}{t < \alpha, \Gamma \rightarrow \Delta, s \in t \oplus \hat{1}}$$

$$C \rightarrow \quad \frac{\rightarrow I(\alpha) ; s \in^* t, \Gamma \rightarrow \Delta}{t < \alpha, s \in t \oplus \hat{1}, \Gamma \rightarrow \Delta}$$

3.3 ORDINAL ASSIGNMENT FOR TC_α

By 3.3.1 we assign constructive ordinals (cf. §3.1.1) to the formulas of TC_α . 3.3.3, 3.3.4, and 3.3.5 together give the assignment of constructive ordinals to the proofs of TC_α by the general method of §2.3, where we have to replace the abstract ordinals and functions by their constructive counterparts.

3.3.1 Definition

We define inductively the complexity of the formulas of TC_α .

The complexity of F will be denoted by $|F|$ or more precisely by $|F|_\alpha$. We distinguish several cases.

(1) F prime

(1.1) F is of the form $t=u$: $|F| = \hat{1}$

(1.2) F is of the form $t \in u$

(1.2.1) t, u are constant terms and $\mathcal{U}((|t|)_1, |u|) \prec \alpha$:

$$|F|_\alpha = \omega(\mathcal{U}((|t|)_1, |u|)) \quad (\text{for } \mathcal{U} \text{ see 3.1.1\#5})$$

(1.2.2) otherwise: $|F|_\alpha = \omega(\alpha)$

(2.1) $|A \wedge B| = |A \vee B| = |A \supset B| = |A \# B|$

(2.2) $|\neg A| = |\forall x A| = |\exists x A| = |A| \# \hat{1}$

3.3.2 Note

If $F \sim F'$, then $|F| = |F'|$.

Proof:

An operation of type 3.2.5#1 on F cannot change $|F|$ since only the values of the terms appear in the definition of $|F|$. For the case 3.2.5#2 we note that $|s \in \mathcal{U}(t, u)|_\alpha = \omega(\mathcal{U}(|t|, |u|))$ if $\mathcal{U}(|t|, |u|) \prec \alpha$ and $= \omega(\alpha)$ otherwise by 3.1.1#5.4, 5.5, which coincides with $|\langle s, t \rangle \in u|_\alpha$.

3.3.3

We define the complexity of the nodes of a derivation in TC_α . If $R(N)$ is a cut then $c(N) = |A|_\alpha$, if it is a TI_β -rule then $c(N) = |A(x)|_\alpha \# \hat{2}$, and for the I -rule we take $|A(x)|_\alpha$, where A is the cut-formula or the induction formula (cf. 2.2.1, 3.2.11). In all other cases $c(N) = \hat{0}$.

3.3.4

As inference functions $\zeta_R(x)$ for 1-rules R we take $x \otimes w(\beta)$ in case of a TI_β , $x \otimes w$ for an I , $x \# \hat{2}$ in case of a C-rule if t in 3.2.12 is not constant, and $x \# \hat{1}$ otherwise. For all 2-rules R , $\zeta_R(x, y) = x \# y$. The conditions in 2.3.1 can easily be verified by 2.1.12.

3.3.5

We assign (cf. 2.3.4) the ordinal $\hat{1}$ to logical axioms and saturated mathematical axioms, and $\hat{2}$ to non-saturated mathematical axioms and C-axioms.

3.3.6

Without loss of generality we may assume that every proof P has the following properties:

- (1) No variable occurs both free and bound in P .
- (2) If a variable is the quantified variable of a quantifier rule or the induction variable (z in 3.2.11) of an induction rule occurring at a node N of P , then it does not occur below N .
- (3) If a variable occurs free in $S(N)$ but not in $S(E)$, then it is the induction variable or quantified variable of a corresponding rule below N .

3.3.7 Lemma

Let P be a proof and x a variable occurring free in some formula of the end-sequent of P . Then by replacing x in P everywhere by a numeral n we get a new proof P_n and $O(P_n) \trianglelefteq O(P)$.

Proof: By 3.3.6 we know that x cannot occur bound in P nor as an induction variable or quantified variable. Then an inspection shows that all rules and axioms remain valid under the replacement of x by n . Hence P_n is a proof. The complexity of the formulas in P (cf. §3.3.1) and thus of the nodes (§3.3.3)

may be lowered by the replacement. But this cannot raise the ordinal of P by 2.4.5. The same can be said about the effects on the ordinals of axioms (§3.3.5) and on the inference function of C -rules (§3.3.4) by 2.4.3. Hence $O(P_n) \leq O(P)$.

3.3.8

By 2.1.12#7 we know that the ordinal of a proof of TC_α for $\alpha \prec \Gamma_0$ is less than Γ_0 . Moreover, we shall show that

$$(1) \quad \alpha^* = \eta(\hat{0}, w(\alpha \hat{\otimes} \hat{1}), \hat{0})$$

is the least upper bound of the ordinals assigned to TC_α .

If $\|x\|$ denotes the classical ordinal corresponding to x ,

then $\|\alpha^*\| = \chi_0(\|\alpha\|+1)$ by §§ 2.1.6, 2.1.11.

Proof:

Note that the function defined by $y(x) = \eta(\hat{0}, x, \hat{0})$ is normal as it corresponds to the classical function y in 2.1.8.

To show that α^* is a bound, it is therefore enough to show that for any proof P of TC_α and any node N there exists $n \prec \omega$ such that $o(P, N) \leq y(w(\alpha) \hat{\otimes} n)$. This is clear if N is topmost since then $o(P, N) \leq \hat{2} \leq y(w(\alpha) \hat{\otimes} \hat{2})$.

If for some n the inequality holds for all predecessors of N , then we can prove that $o^-(N) \leq y(w(\alpha) \hat{\otimes} (n \hat{\otimes} \hat{1}))$ (cf. 2.3.3#3).

Assume e.g. that $R(N)$ is TI_β ($\beta \leq \alpha$). We have $o^-(N) = o(P, N_0) \hat{\otimes} w(\beta) \leq \eta(\hat{0}, w(\alpha) \hat{\otimes} n, \hat{0}) \hat{\otimes} w(\alpha) \leq \eta(\hat{0}, w(\alpha) \hat{\otimes} n, \alpha) \leq y(w(\alpha) \hat{\otimes} (n \hat{\otimes} \hat{1}))$ by

2.1.12#5, 8. If $R(N)$ is another rule we can proceed similarly using 2.1.12#6, 8. Now we can find an $m \prec \omega$ such that for any formula F occurring in P we have $|F|_\alpha \prec w(\alpha) \hat{\otimes} m$ (cf. 3.3.1).

Then $o(P, N) = \eta(h(N), c(N), o^-(N)) \leq \eta(\hat{0}, w(\alpha) \hat{\otimes} m, \eta(\hat{0}, w(\alpha) \hat{\otimes} (n \hat{\otimes} \hat{1}), \hat{0})) = \eta(\hat{0}, w(\alpha) \hat{\otimes} m, \eta(w(\alpha) \hat{\otimes} m, w(\alpha) \hat{\otimes} (m \hat{\otimes} n \hat{\otimes} \hat{1}), \hat{0})) = y(w(\alpha) \hat{\otimes} (n \hat{\otimes} m \hat{\otimes} \hat{1}))$ by 2.1.12#4 and the property of δ in 2.1.11. Hence our statement follows by induction over the tree of P .

We can easily see that there is no $n \prec \omega$ such that

$O(P) \prec y(w(\alpha) \hat{\otimes} n)$ for all proofs of TC_α , so α^* is the least upper bound.

3.4 REDUCTIONS

This section is devoted to the discussion of the reductions for $TC_{\mathcal{N}}$. Compared to number theory, 3.4.3, 3.4.6, and 3.4.7 are new. Some notions used below have been explained in §2.2 and §2.4. We shall be brief, so the reader may find it useful to consult [6], [9], [11].

3.4.1 Definitions

A proof is saturated, if the end-sequent is saturated. By 3.3.6 this is equivalent to saying that $S(N)$ is saturated for every N in P^- .

The principal formula of a C-rule is $\text{set}^{\hat{t}}_1$ in 3.2.12 and set^t is its side formula. In 3.2.11 the formulas shown explicitly in the premiss are side formulas and $A(t)$ is the principal formula. These notions are defined similarly for the logical rules, as usual.

In 3.2.11 and 3.2.12 every formula occurring in Γ or Δ in the conclusion is a successor of the corresponding formula in the premiss. In 3.2.9#2 every A'_i or B'_j is a successor of A_i or B_j , respectively. The successor relation is defined analogously for the other rules, as usual.

The image relation between formulas of a derivation is the transitive and reflexive relation generated naturally by the successor relation, where we consider the occurrence of a formula in $S(N_1)$ and $S_1(N)$ (cf. 2.2.2) as identical. Note that if a formula A occurs at some node N and B is an image of A in a derivation, then B occurs at some $M \sqsubseteq N$.

Furthermore, if M is the predecessor of N and M contains an image of some formula A, while N does not, then A is either a side formula or the (left or right) cut formula at M.

3.4.2 Types of Reductions

We now list six types of reductions. It is understood that the order below determines also the priority according to which reductions have to be applied.

- (1) C-axiom reductions
- (2) thinning reductions
- (3) logical axiom reductions
- (4) induction reductions
- (5) C-fork reductions
- (6) logical fork reductions

Reductions of type (1) are described in 3.4.3. Reductions of types (2) and (3) have been abundantly described in the literature under the heading "preliminary reduction steps". They are treated here as ordinary reductions with one little exception and are discussed briefly in 3.4.4 and 3.4.5.

Type-(4) reductions consist of I-reductions described already in [6] and of TI-reductions defined in 3.4.6. Reductions of type (5), to be treated in 3.4.7, are similar to those of type (6), which are the reduction steps for every logical symbol introduced in [6], [9]. Note that the reductions of type (1),(4),(5) apply only to saturated proofs. We shall stress only the new features, so the reader is referred to the literature for more details.

3.4.3 C-Axiom Reductions

Application condition: P is saturated and a C-axiom is used in P^- . Since P is saturated all terms in the axiom are constant. Now consider e.g. the case #1 in 3.2.8. If $|s| = |u|$ and $|t| = |v|$, then set $\rightarrow uev$ is a mathematical axiom from which #1 is derivable by thinnings. Otherwise $s=u, t=v \rightarrow$ is a mathematical axiom, from which #1 follows again by thinnings. In both cases the ordinal is $\hat{1}$ compared to $\hat{2}$ before. A similar procedure works for cases #2 and #3 in §3.2.8.

3.4.4 Thinning Reductions

We can assume that there is no thinning above E whose thinning formula has an image in the end-sequent, for otherwise we can transform P in finitely many steps so that it conforms to this condition, without changing the tree structure of P nor $O(P)$. Similarly, we can assume that the image of a thinning formula never undergoes a contraction. Now a thinning reduction applies to P if there is a thinning in P^- at some $N \sqsupset E$. Making the mentioned assumption we see that there is a cut at some $M \sqsubset N$ such that the cut formula is an image of the thinning formula. Then it is possible to omit M and the branch above M not containing N . To see that the ordinal is lowered, we can first set $c(M) = \hat{0}$ and apply 2.4.5, then omit the application of ζ in the computation of the ordinal at M , and finally apply 2.4.4 to M .

3.4.5 Logical Axiom Reductions

Such a reduction applies to P if a logical axiom is used in P^- above E . As thinnings have been eliminated it is enough to consider the case where a logical axiom is the premiss of a cut in P^- . But then we can replace this cut and the logical axiom by a conversion, and the ordinal is lowered by the same argument as in 3.4.4.

3.4.6 TI-Reductions

A TI_β -reduction is applicable to a saturated proof P if there is a critical TI_β -rule in P . Assume that this rule occurs at N and $F(N)$ is as in 3.2.11#2. If $|t| \geq \beta$ then $S(N)$ can be derived by thinnings from the axiom $t \prec \beta \rightarrow$ and the ordinal is obviously lowered by this operation. Suppose now $\tau = |t| < \beta$. Then the idea is to replace TI_β by TI_τ . Let P' denote the proof $P \upharpoonright_{N_0}$ of the premiss $z \prec \beta, \forall x \prec z A(x), \Gamma \rightarrow \Delta, A(z)$ and P'_τ be P' with z replaced by τ plus an interchange. We shall give a proof P'' which we can connect with P'_τ to give a new proof of $S(N)$ as follows:

$$\begin{array}{c}
 P'' \\
 \vdots \\
 \hline
 \Gamma \rightarrow \Delta, \forall x \prec \tau A(x) \quad \forall x \prec \tau A(x), \tau \prec \beta, \Gamma \rightarrow \Delta, A(\tau) \\
 \hline
 \tau \prec \beta, \Gamma, \Gamma \rightarrow \Delta, \Delta, A(\tau) \\
 \hline
 \tau \prec \beta, \Gamma \rightarrow \Delta, A(\tau)
 \end{array}
 \quad \text{cut}$$

If $\tau \neq \hat{0}$, P'' is

$$\begin{array}{c}
 P' \\
 \vdots \\
 \hline
 z \prec \tau \rightarrow z \prec \beta \quad z \prec \beta, \forall x \prec z A(x), \Gamma \rightarrow \Delta, A(z) \\
 \hline
 z \prec \tau, \forall x \prec z A(x), \Gamma \rightarrow \Delta, A(z) \\
 \hline
 y \prec \tau, \Gamma \rightarrow \Delta, A(y) \quad \rightarrow \supset \\
 \hline
 \Gamma \rightarrow \Delta, y \prec \tau \supset A(y) \quad \rightarrow \forall \\
 \hline
 \Gamma \rightarrow \Delta, \forall x \prec \tau A(x)
 \end{array}
 \quad \text{cut} \quad TI_\tau$$

If $\tau = \hat{0}$, P'' is

$$\begin{array}{c}
 y \prec \tau \rightarrow \\
 \hline
 y \prec \tau \rightarrow A(y) \quad \rightarrow \supset \\
 \hline
 \rightarrow y \prec \tau \supset A(y) \quad \rightarrow \forall \\
 \hline
 \rightarrow \forall x \prec \tau A(x) \\
 \hline
 \hline
 \Gamma \rightarrow \Delta, \forall x \prec \tau A(x)
 \end{array}$$

A, B, F be $\text{set}\hat{\Theta}1$; $S_P(K) \equiv t \prec \alpha, \Gamma \rightarrow \Delta, \text{set}\hat{\Theta}1$;

$S_P(L) \equiv t \prec \alpha, \text{set}\hat{\Theta}1, \Gamma' \rightarrow \Delta'$; $S_P(M) \equiv \Pi \rightarrow \Sigma$.

We now describe the reduction on P . Note that the term t is constant by assumption (§3.4.1). If $\{t\} \prec \alpha$ is false, then $t \prec \alpha \rightarrow$ is an axiom from which $S(K)$ and $S(L)$ can be derived by thinnings. This simplification clearly lowers the ordinal by 2.4.4.

If $\{t\} \prec \alpha$ is true, then the new proof P^* is going to look like:



$P^* \upharpoonright N'$ is constructed from $P \upharpoonright N$ by replacing the $\rightarrow C$ -rule at K by a thinning with $\text{set}\hat{\Theta}1$ as thinning-formula such that $S_{P^*}(K')$ becomes $t \prec \alpha, \Gamma \rightarrow \Delta, \text{set}^*t, \text{set}\hat{\Theta}1$. The side-formula set^*t of $R_P(K)$ appears now in the sequents below K' in addition to the corresponding ones in P , so that $S_{P^*}(N')$ becomes $\Pi \rightarrow \Sigma, \text{set}^*t$. $P^* \upharpoonright N''$ is constructed by making a similar change with the $C \rightarrow$ -rule at L , such that $S_{P^*}(N'')$ becomes $\text{set}^*t, \Pi \rightarrow \Sigma$. $R_{P^*}(N)$ will be a cut with cut-formula set^*t such that $S_{P^*}(N) \equiv \Pi \rightarrow \Sigma \equiv S_P(N)$. This ends the description of $P^* \upharpoonright N$; the rest of P^* is exactly the same as in P .

We have now to prove that $O(P^*) \prec O(P)$. Let $y = c_P(N)$. Clearly

$c_{P^*}(N') = c_{P^*}(N'') = \nu$. By 3.3.1#1.2.1, we get
 $c_{P^*}(N) = |s \in t|_{\alpha} \leq \omega(|t|) < \omega(|t| \oplus \hat{1}) = |s \in t \oplus \hat{1}|_{\alpha} = c_P(N) = \nu$.
 So $c_{P^*}(N) < c_P(N)$, which is the crucial property of all fork
 reductions. Let $\lambda = h_{P^*}(N)$ and μ be the (ordinal) maximum of
 λ and $c_{P^*}(N)$. Clearly $h_P(N) = \lambda$, $h_{P^*}(N') = h_{P^*}(N'') = \mu$,
 and we have $\lambda \leq \mu < \nu$ by choice of N and the inequality above.
 Since in passing from $P \upharpoonright N$ to $P^* \upharpoonright N'$ we have omitted a strong
 rule at K while the heights remain the same above N , we get by
 the same argument as in 2.4.4 that $o_{P^*}^-(N') < o_P^-(N)$, and similarly
 $o_{P^*}^-(N'') < o_P^-(N)$. Now $o_{P^*}^-(N) = \eta(\mu, \nu, o_{P^*}^-(N')) \# \eta(\mu, \nu, o_{P^*}^-(N'')) <$
 $\eta(\mu, \nu, o_P^-(N))$ by 2.1.12#6. Clearly $o_P(N) = \eta(\lambda, \nu, o_P^-(N))$ and
 $o_{P^*}(N) = \eta(\lambda, c_{P^*}(N), o_{P^*}^-(N))$.
 If $c_{P^*}(N) \leq \lambda$ then $\lambda = \mu < \nu$, so $o_P(N) = o_P^-(N)$ and $o_{P^*}(N) = o_{P^*}^-(N)$
 by 2.1.12#3, whence $o_{P^*}(N) < o_P(N)$. If $c_{P^*}(N) > \lambda$ then $\mu = c_{P^*}(N)$
 and $\lambda < \mu < \nu$, so $o_{P^*}(N) < \eta(\lambda, \mu, \eta(\mu, \nu, o_P^-(N))) = \eta(\lambda, \nu, o_P^-(N)) = o_P(N)$
 by 2.1.12#4,1. Since in both cases $o_{P^*}(N) < o_P(N)$, we can
 conclude $O(P^*) < O(P)$ by 2.4.4.

3.4.8 Remark

Note that the term "reduction" might be misleading, since
 a reduction in our sense need not obviously reduce anything,
 which is most conspicuous in 3.4.7. The only justification
 is the property of lowering the ordinal of the proof to
 which a reduction is applied.

3.5 CONSEQUENCES OF REDUCTION THEORY

In 3.5.1 we state the combinatorial properties of irreducible saturated proofs. An immediate consequence of this lemma is the result on consistency 3.5.2, from which we get a bound for the autonomous ordinals in 3.5.3. We further deduce a result on cut elimination in 3.5.5, which is applied in 3.5.6.

3.5.1 Lemma

Suppose that P is an irreducible saturated proof in TC_α .

If $P^- \neq P$, then there exists a critical C-rule or a critical logical rule whose principal formula has an image in $S_P(E)$.

If $P^- = P$, then $S_P(E)$ can be derived, by thinnings only, from either a logical axiom or a saturated mathematical axiom.

(For the notation, see 2.2.2, 2.2.3, 3.4.1.)

Proof:

Assume $P^- \neq P$. Then there exists at least one node N in P^- with $R(N)$ a C-rule or logical rule, since there cannot be any critical induction rule. For every such N , we show now by induction on the tree underlying P^- that for any $M \subseteq N$, $S(M)$ contains an image of the principal formula of a critical rule (not necessarily of $R(N)$). The statement is clear for $M=N$. Now let $M_1 \subseteq N, M_2$ be the predecessors of $M \subseteq N$, and F be the cut formula of the cut at M . Suppose first there is N' with $M_2 \subseteq N'$ and $R(N')$ critical. By the induction hypothesis, there exist A in $S(M_1)$ and B in $S(M_2)$ which are images of the principal formula of a critical rule. It is impossible that

$A=B=F$ for we would then have the situation of a fork. So an image of A or B is in $S(M)$. If there is no such N' , then it is easy to see that $S(M_2)$ can only contain formulas of the form $t=u$, which is impossible for A . Hence $A \neq F$ and an image of A is in $S(M)$. In both cases, $S(M)$ contains an image of the principal formula of a critical rule, as required. The first part of the lemma now follows by taking $M=E$ above.

Assume that $P^- = P$. It is easy to see that P can only use either saturated mathematical axioms or one logical axiom. But in the first case, $S'_P(E)$ must be a true saturated simple sequent, which we can use as an axiom. The second part of the lemma now follows easily.

3.5.2 Theorem

TC_α is consistent. Moreover, this fact can be proved constructively, using transfinite induction over ordinals in D as the only non-elementary means.

Proof:

There cannot be an irreducible proof of the empty sequent by 3.5.1. The consistency now follows since any proof can be reduced to an irreducible one by 2.4.2. To prove this fact constructively, we introduce first a Gödel numbering of TC_α . Then we define recursive functions ϱ, θ such that if p is the Gödel number of a proof P in TC_α , $\varrho(p)=0$ if P is irreducible and otherwise $\varrho(p)$ gives the Gödel number of some arbitrarily chosen reduction of P , and $\theta(p)=O(P)$.

Assuming that 0 is not the Gödel number of a proof, we define a recursive function σ by means of the fixed point theorem as follows: $\sigma(p)=p$ if $q(p)=0$ and otherwise $\sigma(p) = \sigma(q(p))$. $\sigma(p)$ is obviously the Gödel number of an irreducible proof to which P with Gödel number p reduces. To prove that σ is well-defined, we need induction up to some ordinal α^* which is a bound for the ordinals assigned to proofs in TC_α , as given in 3.3.8. Clearly α^* is still in D .

3.5.3 Theorem

Γ_0 is a bound for the set of autonomous ordinals (cf. §3.2.13).

Proof:

By 2.1.12#7, Γ_0 is a bound for the ordinals assigned to proofs in TC_α for $\alpha < \Gamma_0$. Hence by 3.5.2 we can prove the consistency of the theories $TC_\alpha(\alpha < \Gamma_0)$ by the principle of transfinite induction up to Γ_0 . If we could prove $I(\Gamma_0)$ in some TC_α with $\alpha < \Gamma_0$, then we could formalize the consistency proof in TC_α , which would contradict Gödel's theorem.

Remark

If one does not like this argument using Gödel's theorem, one can also use the standard method [15] §23 using 3.5.5 below.

3.5.4 Definitions

T_α is the system which we get by omitting in TC_α everything which refers to the predicate $\in$. Thus T_α is number theory plus transfinite induction up to α .

TC_α^ω is the (infinitary) system which we get from TC_α by making the following changes:

- (1) The induction rules are replaced by the ω -rule:

If $S(t_1, \dots, t_n)$ has been derived for all n -tuples of constant terms, then $S(x_1, \dots, x_n)$.

- (2) The C-rules obtain the form:

$$\rightarrow C_{\alpha, t} \quad \frac{\Gamma \rightarrow \Delta, s \in^* t}{\Gamma \rightarrow \Delta, s \in t \oplus \hat{1}} \quad C_{\alpha, t} \rightarrow \quad \frac{s \in^* t, \Gamma \rightarrow \Delta}{s \in t \oplus \hat{1}, \Gamma \rightarrow \Delta}$$

where t is constant and $|t| < \alpha$.

- (3) Only axioms 3.2.6 and 3.2.7#1 are used.

3.5.5 Theorem

From a proof P in TC_α we can effectively find a cut free proof P^* in TC_α^ω with the same end-sequent and such that $\tau(P^*) \leq O(P)$.

Proof:

The word "effectively" means that we actually introduce recursive functions in a similar way as in the proof of 3.5.2. Note that the tree ordinal T cannot be recursively represented because of the use of the supremum in the definition, but this is not essential. Now we give the proof intuitively, without referring to Gödel numbers and recursive functions, and proceed by induction on $O(P)$.

Several cases are distinguished.

(1) A reduction is applicable to P .

Then by applying the reduction we get a new proof P' with $O(P') \prec O(P)$. By the induction hypothesis there is a proof P^* in TC_ω^ω with $T(P^*) \preceq O(P')$. P^* has the required properties.

(2) P is irreducible.

(2.1) $P \neq P^-$

(2.1.1) There is a critical C -rule whose principal formula has an image in $S_P(E)$.

Let this rule be $\rightarrow_{C_\alpha} \frac{\Gamma \rightarrow \Delta, u \in *v}{v \prec \alpha, \Gamma \rightarrow \Delta, u \in v \oplus \hat{1}}$. We proceed

similarly in case of a $C \rightarrow_\alpha$. We distinguish three cases.

(2.1.1.1) v contains free variables.

These variables are among the free variables in $S_P(E)$, which are $x_1, \dots, x_n$, say. Let $\bar{t}$ denote the n -tuple of constant terms $(t_1, \dots, t_n)$. Then by substituting the t_i for x_i in P we get a proof $P_{\bar{t}}$ of $S_P(E)_{(t_1, \dots, t_n)}^{(x_1, \dots, x_n)}$ with $O(P_{\bar{t}}) \prec O(P)$ by 3.3.7, 3.3.4, and 2.4.4. By the induction hypothesis there exists a proof $P_{\bar{t}}^*$ in TC_ω^ω with $T(P_{\bar{t}}^*) \preceq O(P_{\bar{t}})$. Joining the proofs $P_{\bar{t}}^*$ by an application of the w -rule, we get a proof P^* of $S_P(E)$ with $T(P^*) \preceq O(P)$, which proves the statement.

(2.1.1.2) v is constant and not $|v| \prec \alpha$.

Then the conclusion of the rule can be derived from the axiom $v \prec \alpha \rightarrow$ by means of thinnings. This operation clearly lowers the ordinal of P and the statement thus follows from the induction hypothesis.

(2.1.1.3) v is constant and $|v| < \alpha$.

Then we replace the critical rule by thinnings

$$\frac{\Gamma \rightarrow \Delta, u \in^* v}{v < \alpha, \Gamma \rightarrow \Delta, u \in v \oplus \hat{1}, u \in^* v},$$

which gives us a proof P' of $\Pi \rightarrow \Sigma, u \in^* v$ if $S_p(E)$ is $\Pi \rightarrow \Sigma$, and we have $O(P') < O(P)$. By the induction hypothesis there is a proof P'' of $\Pi \rightarrow \Sigma, u \in^* v$ in TC_α^w with $T(P'') \leq T(P')$. But since Σ contains an image of $u \in v \oplus \hat{1}$, we can derive $\Pi \rightarrow \Sigma$ from $\Pi \rightarrow \Sigma, u \in^* v$ by applying a $\rightarrow C_{\alpha, t}$ followed by a conversion, a contraction, and some interchanges. This gives us the required proof P^* of $S_p(E)$ in TC_α^w with $T(P^*) \leq O(P)$.

(2.1.2) $S_p(E)$ contains an image of the principal formula of a critical logical rule.

In this case we proceed similarly as in (2.1.1.3).

(2.1.3) otherwise (by 3.5.1) $S_p(E)$ must necessarily contain free variables, and if $P_{\bar{t}}$ is as in (2.1.1.1), $P_{\bar{t}}$ must be reducible to some $P_{\bar{t}}'$ and we have $O(P_{\bar{t}}') < O(P_{\bar{t}}) \leq O(P)$.

Using the induction hypothesis we get proofs $P_{\bar{t}}^*$ in TC_α^w with $T(P_{\bar{t}}^*) < O(P)$ from which we obtain a proof P^* of $S_p(E)$ with $T(P^*) \leq O(P)$ by application of the w -rule.

(2.2) $P = P^-$

(2.2.1) P is saturated.

This case is clear by 3.5.1, for we can get P^* with $T(P^*) = \hat{1}$.

(2.2.2) P is not saturated.

If P consists only of the terminal node with a logical axiom, then the statement is trivial. Otherwise $O(P) \geq \hat{2}$, and the problem can be reduced to (2.2.1) by application of the w -rule.

3.5.6 Theorem

If $\rightarrow F$ is provable in TC_α with F number-theoretic, i.e. not containing the symbol ϵ , then it is provable in T_{α^*} , where α^* is as in §3.3.8.

Proof:

Let $Pr_{TC}(\alpha, x)$ and $Pr_T(\alpha, x)$ denote the provability predicates for TC_α and T_α respectively, and let $Pr^w(\alpha, x)$ denote provability in infinitary number theory with tree ordinal $\prec \alpha$. From a proof of F in TC_α we can construct a cut-free proof of F in TC_α^w by 3.5.5, with a tree ordinal $\prec \alpha^*$ by 3.3.8. This cut-free proof cannot use the predicate ϵ , for otherwise F would also contain ϵ , contrary to our assumption.

By formalizing this argument (which uses transfinite induction up to α^* as the only non-finitary means) we get

$$(1) \quad T_{\alpha^*} \vdash Pr_{TC}(\alpha, \ulcorner \rightarrow F \urcorner) \rightarrow Pr^w(\alpha^*, \ulcorner F \urcorner) .$$

By a technique similar to §3.6.2 we can prove

$$(2) \quad T_{\alpha^*} \vdash Pr^w(\alpha^*, \ulcorner F \urcorner) \rightarrow F .$$

(Note that because of the subformula-principle a partial truth-predicate Tr^n for $n \geq \text{dep}(\ulcorner F \urcorner)$ suffices for the purpose. The argument uses induction on tree ordinals. See also [7].)

Now if $TC_\alpha \vdash \rightarrow F$ then $T_0 \vdash Pr_{TC}(\alpha, \ulcorner \rightarrow F \urcorner)$, since any true closed existential number-theoretic formula is provable in number theory. So $T_{\alpha^*} \vdash \rightarrow F$ by (1) and (2).

3.6 STRENGTH OF TC AND PA

We wish to investigate the strength of the systems TC and PA. 3.6.2 shows that certain reflection principles hold in the hierarchy of systems TC_α . Then we demonstrate in 3.6.3 that Feferman's system of ramified analysis $R [3]$ is interpretable in TC. Finally 3.6.4 determines the set of autonomous ordinals and thus the strength of PA.

3.6.1 Notation and Conventions

Throughout §3.6, our systems TC_α are used in a slightly weaker version than defined in §3.2, which is unessential for our results, since we are concerned with positive results on the strength of TC_α here, but it simplifies the proofs. We make the following changes :

- (1) Logical axioms are of the form $A \rightarrow A$ for some formula A .
- (2) Only a finite set of function symbols is used and therefore only a finite set of mathematical axioms is required.
- (3) C-axioms have the form

$$(3.1) \quad x_2 \leq \alpha, \quad x_1 = x_3, \quad x_2 = x_4, \quad x_1 \in x_2 \rightarrow x_3 \in x_4$$

$$(3.2) \quad x_3 \leq \alpha, \quad \text{Lim}(x_3), \quad \langle x_1, x_2 \rangle \in x_3 \rightarrow x_1 \in \psi(x_2, x_3)$$

$$(3.3) \quad x_3 \leq \alpha, \quad \text{Lim}(x_3), \quad x_1 \in \psi(x_2, x_3) \rightarrow \langle x_1, x_2 \rangle \in x_3$$

- (4) The conversion-rule is omitted.

Next we assume that we are given a Gödel numbering for the systems TC_α . $\text{Var}(x)$, $\text{Trm}(x)$, $\text{Frm}(x)$, $\text{Sqt}(x)$ are recursive predicates which express that x is the Gödel number of a variable, term, formula, sequent, respectively. Prf is a predicate (chosen in a natural way) such that $\text{Prf}(\alpha, \ulcorner p \urcorner, \ulcorner s \urcorner)$

expresses that P is a proof in TC_α with end-sequent S and $Pr(z, x) \leftrightarrow \exists y Prf(z, y, x)$. Terms, formulas, and proofs are formed in a natural way as trees; the maximal length of a path in such a tree is called the depth of the term, formula, or proof and is computed from the Gödel number by the function $dep(x)$. If A is a string of symbols, then $\ulcorner A \urcorner$ is the corresponding Gödel number. The inverse of the function $\ulcorner \urcorner$ is denoted by underlining. This notation is used to express conveniently certain recursive functions, e.g. $\phi(x, y) = \ulcorner x \wedge y \urcorner$ which, if x and y are Gödel numbers of two formulas, has as value the Gödel number of their conjunction. $\bar{x}$ denotes the string consisting of x applications of the symbol for the successor function to the constant 0, so it represents the $(x+1)$ -st numeral. If $Frm(a)$, $Var(w)$, $Trm(t)$ then $\ulcorner a(w/t) \urcorner$ is the Gödel number of the formula constructed from the formula corresponding to a by replacing the variable w by the term t .

We assume further that among the function symbols there are Ψ_T and Ψ_U which allow to express all partial recursive functions by referring to their Gödel numbers similarly as by Kleene's T and U . If ϕ is such a function and $\ulcorner \phi \urcorner$ its Gödel number, then

$A(\phi(x_1, \dots, x_n))$ is an abbreviation for the formula $\exists y [\Psi_T(\ulcorner \phi \urcorner, \langle x_1, \langle x_2, \dots \langle x_{n-1}, x_n \rangle \dots \rangle, y) = 0 \wedge A(\Psi_U(y))]$.

The functions of §3.1.3, §3.6.3 (val, lev, tr) are assumed to be introduced in this way.

3.6.2 Theorem

There is a formula Tr with two free variables such that for any β -formula F with n free variables the following sequents are provable in TC_α if $\beta \oplus \omega \leq \alpha$:

- (1) $\rightarrow \text{Tr}(\beta, \ulcorner \rightarrow F(\bar{x}_1, \dots, \bar{x}_n) \urcorner) \leftrightarrow F(x_1, \dots, x_n)$
- (2) $\text{Pr}(\beta, x) \rightarrow \text{Tr}(\beta, x)$
- (3) $y \leq \beta, \text{Tr}(y, x) \rightarrow \text{Tr}(\beta, x)$

Remark: $\text{Tr}(\beta, \ulcorner \rightarrow F \urcorner)$ can be thought of as expressing that the formula F is true. Note the restriction that F is a β -formula (cf. §3.2.3). We thus cannot derive the liar's paradox from (1), as $\text{Tr}(\beta, \ulcorner \rightarrow F \urcorner)$ will be constructed as a $\beta \oplus \omega$ -formula. The construction is based on the idea that we can speak uniformly over all sets defined arithmetically in C^β by referring to $C^{\beta \oplus \omega}$.

Corollary:

For any α, β with $\beta \oplus \omega \leq \alpha$ and any β -formula F with exactly one variable free we have

- (4) $\text{TC}_\alpha \vdash \text{Pr}(\beta, \ulcorner \rightarrow F(\bar{x}) \urcorner) \rightarrow F(x)$

and the consistency of TC_β is formally provable in TC_α .

Remark:

(4) is a "uniform reflection principle" which clearly implies the "formalized ω -rule"

$$\forall x \text{Pr}(\beta, \ulcorner \rightarrow F(\bar{x}) \urcorner) \rightarrow \forall x F(x) .$$

Thus principles hold in our hierarchy TC_α analogous to those on which Feferman's progressions are based.

Proof of the theorem:

First we give a formal definition of the value of a constant term by induction on the depth:

$\text{val}(\ulcorner 0 \urcorner) = 0, \text{val}(\ulcorner \psi_i(t_1, \dots, t_n) \urcorner) = \psi_i(\text{val}(t_1), \dots, \text{val}(t_n))$ if ψ_i is a function symbol with n arguments. Clearly $\text{val}(\ulcorner \bar{x} \urcorner) = x$ and for any term t with n variables

$$(5) \quad \text{val}(\ulcorner t(\bar{x}_1, \dots, \bar{x}_n) \urcorner) = t(x_1, \dots, x_n) .$$

The level of a formula is defined by $\text{lev}(\ulcorner \underline{s} = \underline{t} \urcorner) = \hat{0}$,

$\text{lev}(\ulcorner \underline{s} \in \underline{t} \urcorner) = \text{val}(t), \text{lev}(\ulcorner \neg A \urcorner) = \text{lev}(\ulcorner \forall x A \urcorner) = \text{lev}(\ulcorner \exists x A \urcorner) = \text{lev}(A),$

$\text{lev}(\ulcorner \underline{A} \wedge \underline{B} \urcorner) = \text{lev}(\ulcorner \underline{A} \vee \underline{B} \urcorner) = \text{lev}(\ulcorner \underline{A} \supset \underline{B} \urcorner) = \text{lev}(A)$ if $\text{lev}(A) \not\geq \text{lev}(B)$

and $= \text{lev}(B)$ if $\text{lev}(B) \geq \text{lev}(A)$. Obviously lev is not defined

for all formulas, but if F is a β -formula, then $\text{lev}(\ulcorner F \urcorner) \leq \beta$.

Now we define truth for prime formulas containing no variables.

$$(6) \quad \text{Tr}^0(z, x) \Leftrightarrow \text{dep}(x) = 0 \wedge \text{frv}(x) = 0 \wedge \exists s \exists t (\text{Trm}(s) \wedge \text{Trm}(t) \wedge \\ \{ \left[x = \ulcorner \underline{s} = \underline{t} \urcorner \wedge \text{val}(s) = \text{val}(t) \right] \vee \\ \left[x = \ulcorner \underline{s} \in \underline{t} \urcorner \wedge \text{val}(s) \in C_{\phi(\text{val}(t), z)}^Z \right] \})$$

where $\phi(y, z) = \phi_3(\phi_0, y, z)$ (cf. §3.1.3). If $\text{lev}(\ulcorner \underline{s} \in \underline{t} \urcorner) \not\leq z$ then

$\text{val}(t) \not\leq z$ and $\text{val}(s) \in C_{\phi(\text{val}(t), z)}^Z \Leftrightarrow \text{val}(s) \in \text{val}(t)$, and this

can be proved in TC_{ω} under the hypothesis $z \leq \omega$. Hence using

(5) we get for any prime formula F with n variables

$$(7) \quad \text{TC}_{\omega} \vdash z \leq \omega, \text{lev}(\ulcorner F(\bar{x}_1, \dots, \bar{x}_n) \urcorner) \leq z \rightarrow \\ \text{Tr}^0(z, \ulcorner F(\bar{x}_1, \dots, \bar{x}_n) \urcorner) \Leftrightarrow F(x_1, \dots, x_n) .$$

We wish to define truth for formulas with depth >0 inductively:

$$(8) \quad \text{Tr}^{n+1}(z, x) \leftrightarrow \text{Tr}^n(z, x) \vee (\text{dep}(x) = n+1 \wedge \text{lev}(x) \leq z \wedge \text{frv}(x) = 0 \wedge \\ \exists a \exists b \exists w (\text{Frm}(a) \wedge \text{Frm}(b) \wedge \text{Var}(w) \wedge \\ \{ [x = \ulcorner a \underline{a} b \urcorner \wedge \text{Tr}^n(z, a) \wedge \text{Tr}^n(z, b)] \vee \\ [x = \ulcorner a \underline{v} b \urcorner \wedge (\text{Tr}^n(z, a) \vee \text{Tr}^n(z, b))] \vee \\ [x = \ulcorner a \underline{> b} \urcorner \wedge (\neg \text{Tr}^n(z, a) \vee \text{Tr}^n(z, b))] \vee \\ [x = \ulcorner \neg a \urcorner \wedge \neg \text{Tr}^n(z, a)] \vee \\ [x = \ulcorner \exists w a \urcorner \wedge \exists y \text{Tr}^n(z, \ulcorner a(\underline{w}/\underline{y}) \urcorner)] \vee \\ [x = \ulcorner \forall w a \urcorner \wedge \forall y \text{Tr}^n(z, \ulcorner a(\underline{w}/\underline{y}) \urcorner)] \}))$$

But this definition is not uniform in n . Therefore we want to define a recursive function tr such that

$$(9.1) \quad \text{TC}_\alpha \vdash z \oplus w \leq \alpha \rightarrow \text{Tr}^n(z, x) \leftrightarrow x \in C_{\text{tr}(z, n)}^{z \oplus \hat{2}n}$$

for every n . This is easily accomplished by induction on n applying 3.1.3 to (6) and (8) (tacit use of the recursion theorem), since if Tr^n is $\sum_1(C^y)$ then Tr^{n+1} is $\sum_3(C^y)$ and thus $\sum_1(C^{y \oplus \hat{2}})$. By 3.1.3#3 we can define tr^* such that

$$(9.2) \quad \text{TC}_\alpha \vdash z \oplus w \leq \alpha \rightarrow x \in C_{\text{tr}(z, n)}^{z \oplus \hat{2}n} \leftrightarrow x \in C_{\text{tr}^*(z, n)}^{z \oplus w}$$

$C_{\text{tr}^*(z, n)}^{z \oplus w}$ can be considered as the set of Gödel numbers of closed formulas with depth $\leq n$ which are true "with respect to C^z ". $x \in C_{\text{tr}^*(z, \text{dep}(x))}^{z \oplus w}$ will then be the universal truth-definition for closed formulas. The formula Tr is so chosen that it expresses this relation, i.e.

$$(9.3) \quad \text{TC}_\alpha \vdash z \in D, \text{Frm}(x), \text{frv}(x) = 0 \rightarrow \text{Tr}(z, x) \leftrightarrow x \in C_{\text{tr}^*(z, \text{dep}(x))}^{z \oplus w}$$

For the cases where $\text{frv}(x) > 0$ or $\text{Sq}(x)$, we make Tr express the following. A formula is true if it is true for all assignments of values to its free variables. A sequent is

true if for all assignments of values to its free variables there is a true formula in the succedent or a non-true formula in the antecedent. Note that to speak about all assignments it is sufficient to have a coding for all finite sequences, which is fulfilled already in ordinary number theory. We omit details of formalization, but only state that for any formula A

$$(10) \quad \text{Tr}(z, \ulcorner A \urcorner) \longleftrightarrow \text{Tr}(z, \ulcorner A \urcorner) \longleftrightarrow \forall x \text{Tr}(z, \ulcorner A(x/\bar{x}) \urcorner) .$$

From (8) and (9) we can derive for $1 \leq i \leq 4$, $5 \leq j \leq 6$

$$(11) \quad \text{TC}_\alpha \vdash z \oplus w \leq \alpha, \text{Frm}(x), \text{Frm}(y), \text{frv}(x)=0, \text{frv}(y)=0 \rightarrow (11.i)$$

$$\text{TC}_\alpha \vdash z \oplus w \leq \alpha, \text{Frm}(x), \text{Var}(w), \text{frv}(\ulcorner \underline{x}(\underline{w}/\bar{y}) \urcorner)=0 \rightarrow (11.j)$$

$$(11.1) \quad \text{Tr}(z, \ulcorner \underline{x} \wedge \underline{y} \urcorner) \longleftrightarrow \text{Tr}(z, x) \wedge \text{Tr}(z, y)$$

$$(11.2) \quad \text{Tr}(z, \ulcorner \underline{x} \vee \underline{y} \urcorner) \longleftrightarrow \text{Tr}(z, x) \vee \text{Tr}(z, y)$$

$$(11.3) \quad \text{Tr}(z, \ulcorner \underline{x} \supset \underline{y} \urcorner) \longleftrightarrow \text{Tr}(z, x) \supset \text{Tr}(z, y)$$

$$(11.4) \quad \text{Tr}(z, \ulcorner \neg \underline{x} \urcorner) \longleftrightarrow \neg \text{Tr}(z, x)$$

$$(11.5) \quad \text{Tr}(z, \ulcorner \exists \underline{w} \underline{x} \urcorner) \longleftrightarrow \exists y \text{Tr}(z, \ulcorner \underline{x}(\underline{w}/\bar{y}) \urcorner)$$

$$(11.6) \quad \text{Tr}(z, \ulcorner \forall \underline{w} \underline{x} \urcorner) \longleftrightarrow \forall y \text{Tr}(z, \ulcorner \underline{x}(\underline{w}/\bar{y}) \urcorner)$$

By induction on the depth of formulas, we can derive from (7) and (11) that for any formula F with exactly n free variables

$$(12) \quad \text{TC}_\alpha \vdash \text{lev}(\ulcorner F(\bar{x}_1, \dots, \bar{x}_n) \urcorner) \leq z, z \oplus w \leq \alpha \rightarrow \\ \text{Tr}(z, \ulcorner F(\bar{x}_1, \dots, \bar{x}_n) \urcorner) \longleftrightarrow F(x_1, \dots, x_n)$$

which clearly implies (1).

Next we want to prove

$$(13) \quad \text{TC}_\alpha \vdash z \oplus w \leq \alpha, \text{Pr}(z, x) \rightarrow \text{Tr}(z, x)$$

i.e. soundness of TC with respect to our truth definition.

We use induction on the depth of proofs, i.e. we have to

show under the hypothesis $z \oplus \omega \leq \alpha$ that $Ax(z, x) \supset Tr(z, x)$ and $Tr(z, y) \wedge In_R(z, y, x) \supset Tr(z, x)$ for any 1-rule R (and correspondingly for 2-rules), where $Ax(\beta, \ulcorner S \urcorner)$ means that the sequent S is an axiom of TC_β and $In_R(\beta, \ulcorner S \urcorner, \ulcorner S^* \urcorner)$ means that S^* is derivable from S by application of R in TC_β .

As for the axioms of the form $A \rightarrow A$, it is clear that $Frm(x) \supset Tr(z, \ulcorner \underline{x} \rightarrow \underline{x} \urcorner)$ and if S is one of the (finitely many) other axioms (cf. 3.6.1), $Tr(z, \ulcorner S \urcorner)$ is derivable from the corresponding axiom in TC_α using (12). The logical rules are treated by (11). For the quantifier rules, note in addition that $val(s) = val(t) \supset Tr(z, \ulcorner \underline{a}(\underline{s}) \urcorner) \leftrightarrow Tr(z, \ulcorner \underline{a}(\underline{t}) \urcorner)$ and $frv(t) = 0 \supset val(t) = val(\ulcorner \overline{val(t)} \urcorner)$ is provable in TC_α under the hypothesis $Frm(a)$, $Trm(s)$, $Trm(t)$, $z \oplus \omega \leq \alpha$. The induction rules are treated by using the corresponding rules in TC_α and (11), the C-rules by the corresponding rules in TC_α and (12). Clearly (13) implies (2) and (3) follows from the fact that the implications $Tr(y, x) \supset lev(x) \leq y$ and $lev(x) \leq y \leq z \supset Tr(y, x) \leftrightarrow Tr(z, x)$ hold.

3.6.3 Theorem

Feferman's ramified analysis R_α (cf. [3]) is interpretable in TC_α where $\tilde{x} = \omega \oplus (\hat{1} \oplus x)$, i.e. we can give a translation F^* to every formula F of R_α such that if $R_\alpha \vdash F$ then $TC_\alpha \vdash F^*$.

Remark:

We assume that Feferman's hierarchy of systems R_α is based on our system of ordinals D (§3.1.1) and the same symbols for primitive recursive functions are used as in TC .

The idea of the interpretation is to let the set variables of level α in R to range over the sets of level less than $\tilde{\alpha}$ in the sense of §3.1.2. This is closely related to Kleene's result cited in 4.7 of [3] .

The essential axioms of R_α are the following, where $\text{Pr}_R(\beta, \ulcorner F \urcorner)$ expresses $R_\beta \vdash F$ and $d(F)$ denotes the degree of F , i.e. the maximum of all α such that S^α occurs free in F , and of all $\beta \hat{\oplus} 1$, such that S^β occurs bound in F :

- (1) RC_β : $\exists S^\beta \forall x (x \in S^\beta \leftrightarrow F(x))$
where $\beta \leq \alpha$, $d(F) \leq \beta$, S not free in F .
- (2) OR_β : $\forall x \text{Pr}_R(\beta, \ulcorner F(\bar{x}) \urcorner) \supset \forall x F(x)$
where $\beta < \alpha$, F a formula with just x free.
- (3) $\text{LG}_{\beta\gamma}$: $\forall x \prec \gamma \text{Pr}_R(\beta, \ulcorner \forall S^\gamma F(S^\gamma) \urcorner) \supset \forall S^\gamma F(S^\gamma)$
where $\gamma \leq \beta < \alpha$, $\text{Lim}(\gamma)$, F a formula with just S free.

RC is the (ramified) comprehension axiom, OR is a formalized ω -rule, and LG is the "limit generalization rule".

We wish to present the idea behind the proof. First remark that $\tilde{x}$ is a normal function and $\tilde{x} \hat{\oplus} \omega = \widetilde{x \hat{\oplus} 1}$. To see that the interpretation of (1) is true, note that a set arithmetical in some set of level $\prec \tilde{\alpha}$ is again a set of level $\prec \tilde{\alpha}$ (by 3.1.3, since $\tilde{\alpha}$ is a limit number) and that quantification over all sets of level $\prec \tilde{\beta}$ can be reduced to quantification over elements of $C^{\tilde{\beta}}$. For axiom (3) remark that the class of sets of level $\prec \tilde{\gamma}$ is identical to the union of the classes of sets of level $\prec \tilde{x}$ for any $x \prec \gamma$. (2) and (3) have the character of

reflection principles. To cope with them, we need within TC_{α} a model for all R_{β} with $\beta \leq \alpha$, which can be built by means of 3.6.2. In fact, such a model could also be given for R_{α} , but this is not important for us. Note that by a model for a system S_1 within a system S_2 we understand here an actual truth-definition for the formulas of S_1 , which allows a formal consistency proof for S_1 in S_2 , whereas the properties of a model formed by an interpretation can in general be seen only from the outside.

Proof of the theorem:

First we take an injective mapping from the set of all variables of R into that of TC . The image of a set variable S^{β} under this mapping will be denoted by s^{β} or s , while the same notation will be used for number variables and their images.

Let $x \in^Z y$ be an abbreviation for the formula $\langle \langle x, (y)_0 \rangle, (y)_1 \rangle \in \tilde{Z}$.

Then by 3.1.2, 3.1.1#5, and $\text{Lim}(\tilde{Z})$ we have

$$(4) \quad x \in^Z \langle e, n \rangle \leftrightarrow x \in C_e^{\pi(\varphi(n, \tilde{Z}))}$$

So y in $x \in^Z y$ may be thought to be a variable for sets of level $\omega \tilde{Z}$, as $\pi(\varphi(n, \tilde{Z}))$ is a fundamental sequence for $\tilde{Z}$.

Now let the translation F^* be constructed from F by first replacing every variable by its image and then every prime formula of the form $t \in s^{\beta}$ occurring in the result by $t \in^{\beta} s^{\beta}$.

The logical structure of F^* is clearly the same as of F , and we shall not make a notational difference between F and F^* if F is a first-order formula.

To see that the interpretation given by the translation just defined is correct we proceed by induction on the length of the proofs in $R_{\mathcal{L}}$. We confine ourselves to the essential steps, which are to prove $TC_{\mathcal{L}} \vdash \rightarrow F^*$ if F is one of the axioms (1), (2), (3).

To treat case (1), we first show inductively that F^* is equivalent in $TC_{\mathcal{L}}$ to a formula of the form $t_F \in^{\beta} \langle v_F, w_F \rangle$ with suitable terms t, v, w , using 3.1.3 and (4). If F has the form $s=u$, then $F^* \equiv F$ and we can take $t_F \equiv \langle s, u \rangle$, $v_F \equiv \phi$, and $w_F \equiv 0$. If F has the form $t \in Z^{\lambda}$, we can proceed as follows. F^* is $t \in^{\lambda} z$ and we know that $\lambda \leq \beta \leq \alpha$. We set $t_F \equiv t$. If $\lambda = \beta$ we choose $v_F \equiv (z)_0$, $w_F \equiv (z)_1$. If $\lambda < \beta$ then $\tilde{\lambda} < \tilde{\beta}$ and by 3.1.1#5 $\pi(\varphi((z)_1, \tilde{\gamma})) < \tilde{\gamma} \leq \pi(\varphi(\partial(\tilde{\gamma}, \tilde{\beta}), \tilde{\beta}))$. So we may choose $w_F \equiv \partial(\tilde{\gamma}, \tilde{\beta})$, $v_F \equiv \phi_3((z)_0, \pi(\varphi((z)_1, \tilde{\gamma})), \pi(\varphi(w_F, \tilde{\beta})))$. For the induction step, if F is composite, we use 3.1.3##3, 5...9. So we can take $w_{A \wedge B} \equiv w_{A \vee B} \equiv \max(w_A, w_B)$, $w_{\neg A} \equiv w_A + 1$. $A \supset B$ is treated by the equivalent $\neg A \vee B$. By 3.1.3##1, 2 we can define ϕ such that for any term u with n free variables

$$(5) \quad u(x_1, \dots, x_n) \in C_e^m \Leftrightarrow x_1 \in C_{\phi}^m(\ulcorner u \urcorner, \langle x_2, \langle \dots \langle x_{n-1}, x_n \rangle \dots \rangle \rangle, e)$$

For the cases $F \equiv \forall y A$ and $F \equiv \exists y A$ we have to use (5) first, since the image of y may appear in t_A , whereas if $F \equiv \forall s^{\lambda} A$ or $F \equiv \exists s^{\lambda} A$ the image variable s^{λ} can only occur in v_A since $d(F) \leq \beta$ implies $\lambda < \beta$. We can take $w_{\forall y A} \equiv w_{\forall s^{\lambda} A} \equiv w_A + 2$, $w_{\exists y A} \equiv w_{\exists s^{\lambda} A} \equiv w_A$. We leave it to the reader to find t_F and v_F for these cases. Using (5) we finally obtain a formula $x \in^{\beta} u$ which is equivalent to $F^*(x)$, where u is a term not containing x . Hence

$TC_{\alpha} \vdash \exists s \forall x (x \in^{\beta} s \leftrightarrow F^*(x))$, i.e. the translation of (1) is provable in TC_{α} .

Let τ be a recursive function such that $\tau(\ulcorner F \urcorner) = \ulcorner F^* \urcorner$. Clearly

$$(6) \quad Pr_R(z, x) \supset lev(\tau(x)) \leq z.$$

We assume for a moment that

$$(7) \quad TC_{\alpha} \vdash Pr_R(\beta, x) \rightarrow Tr(\tilde{\beta}, \tau(x)) \quad \text{if } \beta < \alpha,$$

which expresses formally the correctness of the interpretation of R_{β} or the soundness of R_{β} with respect to the model formed in TC_{α} by $Tr(\tilde{\beta}, \tau(x))$.

Now let α, β, F be as in (2), so $\beta \oplus \omega \leq \alpha$. Then by (7)

$Pr_R(\beta, \ulcorner F(\bar{x}) \urcorner) \rightarrow Tr(\tilde{\beta}, \ulcorner F^*(\bar{x}) \urcorner)$ is provable in TC_{α} . But $TC_{\alpha} \vdash Tr(\tilde{\beta}, \ulcorner F^*(\bar{x}) \urcorner) \rightarrow F^*(x)$ by 3.6.2#1 since F^* is a $\tilde{\beta}$ -formula. Hence $TC_{\alpha} \vdash \forall x Pr_R(\beta, \ulcorner F(\bar{x}) \urcorner) \rightarrow \forall x F^*(x)$ which shows that $TC_{\alpha} \vdash (OR_{\beta})^*$.

Let α, β, γ, F be as in (3). Then by (7)

$$TC_{\alpha} \vdash Pr_R(\beta, \ulcorner \forall s \bar{x} F(\epsilon s \bar{x}) \urcorner) \rightarrow Tr(\tilde{\beta}, \ulcorner \forall s \bar{x} F^*(\epsilon \bar{x} s \bar{x}) \urcorner).$$

After replacing in F^* the variables s^x by some new variable s , which does not change the truth value, and then applying (6) and 3.6.2#12 we get $TC_{\alpha} \vdash Pr_R(\beta, \ulcorner \forall s \bar{x} F(\epsilon s \bar{x}) \urcorner) \rightarrow \forall s F^*(\epsilon^x s)$. But $TC_{\alpha} \vdash \forall x < \gamma \quad \forall s F^*(\epsilon^x s) \leftrightarrow \forall s F^*(\epsilon^{\gamma} s)$, as for any e, n there are $x < \gamma, f, m$ such that $y \in^{\gamma} \langle e, n \rangle \leftrightarrow y \in C_e^{\pi}(\gamma(n, \tilde{\gamma})) \leftrightarrow y \in C_f^{\pi}(\gamma(m, \tilde{x})) \leftrightarrow y \in^x \langle f, m \rangle$ and conversely (use (4), the normality of the function $\tilde{x}$, 3.1.3#3, 3.1.1#5). Hence

$TC_{\alpha} \vdash \forall x < \gamma Pr_R(\beta, \ulcorner \forall s \bar{x} F(\epsilon s \bar{x}) \urcorner) \supset \forall s F^*(\epsilon^{\gamma} s)$ which shows

$$TC_{\alpha} \vdash (LG_{\beta \gamma})^*.$$

It remains to prove (7). This is done by proving

$$(8) \quad \rightarrow \forall z \leq \alpha \forall x \text{Pr}_R(z, x) \supset \text{Tr}(\tilde{z}, \tau(x))$$

in $\text{TC}_{\omega}^{\omega}$ by transfinite induction on z and induction on the depth of the proofs in R . The treatment of axioms and rules of pure number theory follows the proof of 3.6.2#13, because the translation of the formulas does not change the logical structure (only second-order variables are mapped into first-order). Axioms (1), (2), (3) are treated by a formalization of the arguments above, where we use the (transfinite) induction hypothesis instead of assumption (7).

We take case (2) as an example, arguing in $\text{TC}_{\omega}^{\omega}$.

By the induction hypothesis $\forall z \leq \alpha \forall y \leq z \forall x [\text{Pr}_R(y, x) \supset \text{Tr}(\tilde{y}, \tau(x))]$ we

$$\text{get } \forall z \leq \alpha \forall y \leq z \forall f_y [\forall x \text{Pr}_R(y, \ulcorner \underline{f}(\bar{x}) \urcorner) \supset \forall x \text{Tr}(\tilde{y}, \ulcorner \underline{\tau(f)}(\bar{x}) \urcorner)],$$

where $\forall f_y$ denotes quantification restricted to Gödel numbers of

formulas of R with one free number variable and set variables

of level $\leq y$ only. Now $y \leq z \supset \text{lev}(\ulcorner \forall x \text{Tr}(\tilde{y}, \ulcorner \underline{\tau(\bar{f})}(\bar{x}) \urcorner) \urcorner) = \tilde{y} \oplus \omega \leq \tilde{z}$

and $\text{lev}(\ulcorner \text{Pr}_R(\tilde{y}, \ulcorner \underline{\bar{f}}(\bar{x}) \urcorner) \urcorner) = \hat{0}$, so by 3.6.2 #12 we get

$$(9) \quad \forall z \leq \alpha \forall y \leq z \forall f_y [\text{Tr}(\tilde{z}, \ulcorner \forall x \text{Pr}_R(\tilde{y}, \ulcorner \underline{\bar{f}}(\bar{x}) \urcorner) \urcorner) \supset \text{Tr}(\tilde{z}, \ulcorner \forall x \text{Tr}(\tilde{y}, \ulcorner \underline{\tau(\bar{f})}(\bar{x}) \urcorner) \urcorner)].$$

But a formalization of the (finitary) argument treating case (2)

$$\text{above leads to } \forall y \leq z \forall f_y \text{Pr}(\tilde{z}, \ulcorner \forall x \text{Tr}(\tilde{y}, \ulcorner \underline{\tau(\bar{f})}(\bar{x}) \urcorner) \urcorner) \supset \forall x \text{Tr}(\tilde{z}, \ulcorner \underline{\tau(f)}(x) \urcorner)$$

from which we get by 3.6.2#13, 11.3

$$\forall z \leq \alpha \forall y \leq z \forall f_y [\text{Tr}(\tilde{z}, \ulcorner \forall x \text{Tr}(\tilde{y}, \ulcorner \underline{\tau(\bar{f})}(\bar{x}) \urcorner) \urcorner) \supset \text{Tr}(\tilde{z}, \ulcorner \forall x \underline{\tau(f)}(x) \urcorner)].$$

Hence by (9) and 3.6.2#11.3

$$\forall z \leq \alpha \forall y \leq z \forall f_y \text{Tr}(\tilde{z}, \ulcorner \forall x \text{Pr}_R(\tilde{y}, \ulcorner \underline{\bar{f}}(\bar{x}) \urcorner) \urcorner) \supset \forall x \underline{\tau(f)}(x)$$

which proves (8) for the case that x is an OR-axiom, since

$$\tau(\ulcorner \forall x \text{Pr}_R(\tilde{y}, \ulcorner \underline{\bar{f}}(\bar{x}) \urcorner) \urcorner) \supset \forall x \underline{\tau(f)}(x) = \ulcorner \forall x \text{Pr}_R(\tilde{y}, \ulcorner \underline{\bar{f}}(\bar{x}) \urcorner) \urcorner \supset \forall x \underline{\tau(f)}(x).$$

3.6.4 Theorem

The set A of autonomous ordinals can be characterized by

$$A = \{x \in D \mid x \prec \Gamma_0\}.$$

Proof:

We prove by transfinite induction that every ordinal autonomous with respect to R is also autonomous with respect to TC . If α is autonomous with respect to R , then we must have been able to show $R_\beta \vdash I_\gamma(\alpha)$ for some $\gamma \leq \beta \prec \alpha$, so $TC_\beta \vdash \rightarrow (I_\gamma(\alpha))^*$ by 3.6.3 (cf. [3] for the notation I_γ). By the induction hypothesis, β is autonomous with respect to TC and so is $\tilde{\beta}$, since we can easily prove $TC_\beta \vdash \rightarrow I(\tilde{\beta})$. By the definition of the translation $*$ the implication $(I_\gamma(\alpha))^* \supset I(\alpha)$ holds and thus α is autonomous with respect to TC . (The use of transfinite induction in this argument may be avoided.) We conclude that A must contain at least all ordinals $\prec \Gamma_0$, by the result of Feferman [3] (6.24). The theorem now follows by 3.5.3.

3.6.5 Theorem

The system of predicative analysis PA is a conservative extension of first-order number-theory plus transfinite induction up to any ordinal $\prec \Gamma_0$.

Proof

immediate by 3.5.6 and 3.6.4.

4 CONCLUSION

4.1

We have shown that the "old-fashioned" Gentzen technique is well suited for treating predicative analysis. By stressing the basic ideas in §2 we have made an attempt to diminish the general opinion that Gentzen type proof theory is "complicated". At any rate it does not seem to be more difficult than the developments in Schütte [16] and Tait [18]. Furthermore, the advantage is here that the technique is applied immediately to a simple and intuitively appealing formal system, which gives a concise treatment of predicative analysis. We use infinitary systems only where the subformula principle is essential and thus cut elimination is required. Remember also that the definition of a system of our hierarchy TC does not use a reflection principle on a theory with lower level (ordinal), as is the case for Feferman's progressions of theories; however by §3.6.2 reflection principles hold within our hierarchy, too. It may seem strange to define the system PA of predicative analysis as a first-order system, but an inspection of §3.6.3 shows that a simple definitional extension would allow us to speak easily about the sets of ramified analysis. It would also be possible to give an interpretation of Schütte's system [17] or Feferman's IR into PA. The essential feature of IR (cf. [3] 6.28 iii) is the definition of sets by an arbitrary transfinite recursion from an arbitrary set compared to the use of

a special hierarchy in PA. Schütte's system has variable levels in common with PA.

4.2

We add some remarks on the limitations of Gentzen's technique. Note that the reductions of the type "elimination of a fork" depend essentially on the fact that the side formulas of a given rule have lower complexity than the principal formula. But in the case of 2nd order rules this may fail because $\forall X A(X)$ has not in general a higher complexity than $A(T)$, unless we have a bound for the complexities of the set terms T . So it is clear that this method breaks down if we have essential impredicativities in a theory, such as Π_1^1 -analysis. Two other approaches lead to the same conclusion. Kreisel remarks in [7] that every theorem of a theory with recursive proof trees and a proper subformula principle can be proved by transfinite induction on a recursive well-ordering. He concludes that Gentzen's technique must stop before Π_1^1 -analysis. The same result is obtained by Diana Schmidt, who shows that certain ordinals cannot be reached by the use of monotonic functions (which are used in the assignment of ordinals). So Takeuti's proof theory [19] is much stronger than Gentzen's. Note however that the intuitionistic systems considered by Scarpellini in [11], [12], [13] are so strong that cut elimination (even after addition of an ω -rule) does not hold. In fact, reductions can be applied only to proofs of a special form, which suffices for the solution of the problems studied there.

4.3

It would be interesting to search for applications of Gentzen's technique to other systems or other problems, keeping in mind §4.2. Furthermore we should perform the proofs of well-ordering directly in our formal systems using the ideas of [4] and [17]. This together with an improved assignment of ordinals (if necessary) would also determine the exact bound for the provable ordinals of TC_{ω} .

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CURRICULUM VITAE

Am 7.Juni 1946 wurde ich, Roger Schoenberger, Bürger von Basel, als Sohn des kaufmännischen Adjunkten Charles Schoenberger und seiner Frau Paula, geb. Mariani, in Basel geboren. Ich durchlief das Realgymnasium Basel, das ich 1965 mit der Matur Typ B verliess. Anschliessend begann ich das Studium der Physik und Mathematik an der Universität Basel. 1969 bestand ich das Diplom in theoretischer Physik unter Prof. K.Alder. 1970 begann ich das Studium in mathematischer Logik und die Ausarbeitung der vorliegenden Dissertation unter Prof. B.Scarpellini. Von April 1970 bis März 1973 war ich als Assistent am Mathematischen Institut der Universität Basel angestellt.

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